MATHEMATICS
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MATHEMATICS

BOOK I

THE CERTIFICATE LIBRARY

THIS SERIES of books has been designed to cover the various syllabuses of the Ordinary Level Examination for the General Certificate of Education. The books are equally suitable for all students who wish to improve their standards of learning in the subjects of the various volumes, and together they form an invaluable reference library.

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Each volume is comprehensive and self-contained and therefore has far more than an ordinary class textbook: each has a full treatment of the subject; numerous illustrations; questions on the chapters with answers at the end of the book; and a Revision Summary for each section of the book.

These summaries give the information in a concise and compact way so that examination candidates may easily revise the whole contents of the book—they can soon see how much they readily know, and how much they have forgotten and need to study again. The facts in these summaries serve, therefore, as a series of major pegs upon which the whole fabric of the book hangs.

Advice is given on answering the examination paper and there are questions of examination standard together with suggested answers.

Series Executive Editor
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MATHEMATICS

BOOK 1

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PREFACE

THIS IS the first of two volumes devoted to Mathematics and will take you half-way towards your O level G.C.E. in that subject. It will need co-operation from you, of course, for mathematics is a subject to be learned rather than taught.

What is your attitude to mathematics? If you "know the rule," and when to apply it, are you quite satisfied? Do you never bother about where the particular rule "came from," and why it works? Can you add, subtract, multiply and divide without any idea why the processes you use give the right answer? Do you say, "Give me the rules, and I will finish the job," and feel that the rules themselves are not your business? If you do feel like this—and many people do—then you must face the fact that your attitude is *quite wrong*, and is standing in the way of your mathematical progress.

Mathematics is a sensible, logical subject, with rules which should be clear and obvious to anyone learning it. But unless you want to understand the reason for each process, and why it makes sense, you are not approaching the subject in the right way, and you will soon involve yourself in needless difficulties because you have no insight into what is going on.

While working through the book, do at all times try to understand why the processes work; don't content yourself with memorizing them parrot-fashion. If you can't see the sense in some process, then read the explanation again, think about it, ask somebody, but don't just give up. Be determined, all the time, to *understand*. This attitude is half the battle.

This book is divided into separate sections—Arithmetic, Algebra and Geometry—because it is easier to look up and to follow through material arranged in this way—*not* because these subjects are in separate compartments and not to be confused with each other! In fact, several of the examples bring in more than one branch of mathematics; and there is no objection to solving arithmetical or geometrical problems by algebra, or algebraic problems by arithmetic—quite the reverse, in fact. Use whatever seem to you the best tools for the job.

The brief Revision Summaries will help you to check quickly whether you have really mastered the work. If you have, you will be ready for Book 2.

H. R. CHILLINGWORTH

Note to 1971 *edition.* In this revision decimal currency has been used for examples and sums involving amounts of money; and the new standard abbreviations have been used for examples and questions involving metric or Imperial units.

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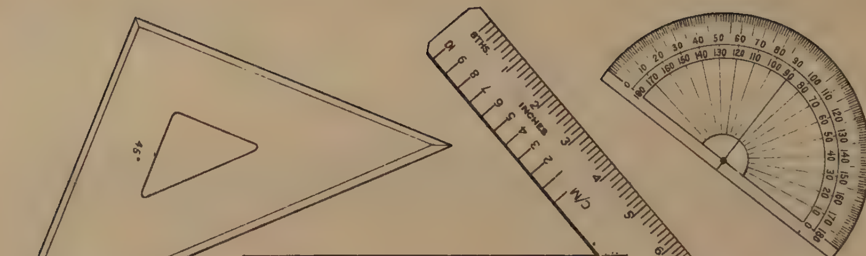
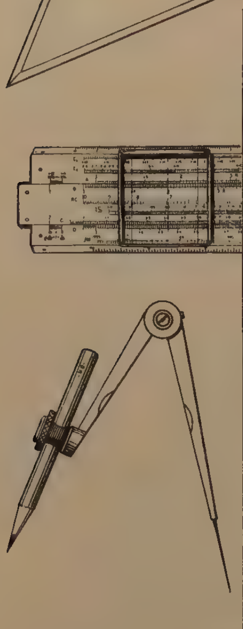
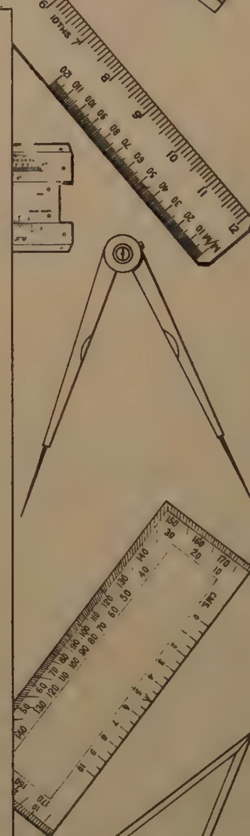
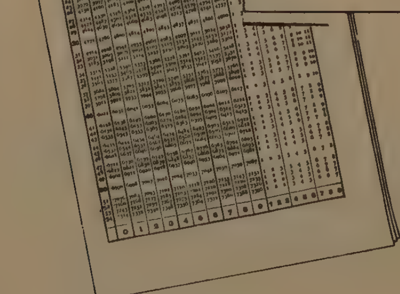


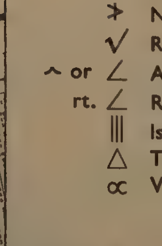


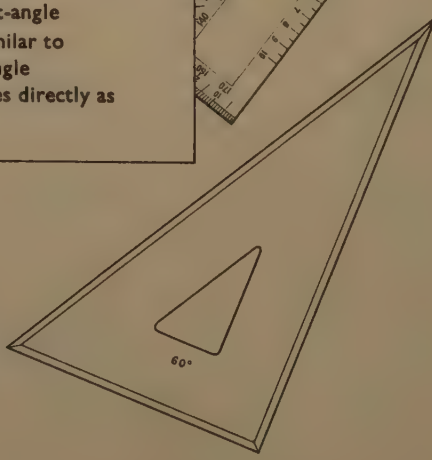


Mathematical symbols and their meaning

$\therefore$	Therefore
$\because$	Because
$:$	Ratio
$+$	Plus
$-$	Minus
$\pm$	Plus or Minus
$\times$	Multiplied by
$\div$	Divided by
$=$	Equal to
$\neq$	Not equal to
$\bumpeq$	Approx. equal to
$\equiv$	Identical with; Congruent
$\perp$	Perpendicular to
$<$	Less than
$\nless$	Not less than
$>$	More than
$\ngtr$	Not more than
$\surd$	Radical (root)
$\frown$ or $\angle$	Angle
rt. $\angle$	Right-angle
$\vert\vert\vert$	Is similar to
$\triangle$	Triangle
$\propto$	Varies directly as







INTRODUCTION

THE WRITING OF UNITS

Money

BRITISH CURRENCY is based on the *pound*, for which the symbol is £, and amounts of money in whole numbers of pounds are written with this symbol first, for example, two hundred pounds should be written as

£200

For smaller amounts the pound is divided into 100 *pence*, for which the symbol is p and this symbol is written after the number; for example, forty and a half pence is written

$40\frac{1}{2}$p

To express a mixture of pounds and pence, *only the £ symbol is used*, and the numbers of pounds and pence are separated by a decimal point:

£3·20

symbolises three pounds and twenty pence. *Never use both £ and p in the same expression.* Note, too, that the decimal point should always be followed by *two* digits; thus three pounds and two pence is written

£3·02

An odd halfpenny is shown as

£2·07$\frac{1}{2}$

The new decimal coins.

For some purposes it is useful to employ the £ symbol when writing amounts less than a pound; for example, a long list of sums of money might contain just one or two amounts less than a pound, and a change of symbol could be confusing. In such a case we *always* show that the amount contains no pounds by writing a zero symbol before the decimal point, thus:

£0·28 means 28 pence.

Measures

There is an internationally agreed system of units known as SI (Système International d'Unités) which is based on the metric system; this will be discussed in Chapter 6. This system is in use in Great Britain for most scientific and engineering purposes, and its use will become more and more general in the next few years. For the moment, however, the Imperial system will continue to be used in everyday life, and it is used quite a lot in this book. Here is a list of the symbols used for the various units:

THE WRITING OF UNITS

pint	pt	inch	in	ounce	oz	second	s
quart	qt	foot	ft	pound	lb	minute	min
gallon	gal	yard	yd	hundredweight	cwt	hour	h
		mile	mile				

There are two important things to remember when using these symbols:

(a) A full stop is NOT put after the symbol (unless, of course, it is to indicate the end of a sentence).

(b) The letter *s* is NOT added to the symbol to indicate a plural.

Two units are often combined to indicate a *rate* (see Chapter 3). In such circumstances the word *per* often appears, as in the phrases *miles per hour* or *gallons per minute*. In writing such expressions the oblique stroke / is used to represent the word *per*, so that, for example, the phrase

50 feet per second

is written as

50 ft/s

There are other ways of writing rates, but they are not used in this book.

ARITHMETIC

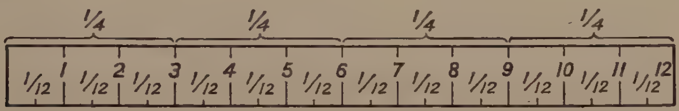
CHAPTER 1

FRACTIONS I

THE WORD *fraction* comes from a Latin word meaning *break*. (If you break your arm, the doctor will call it a fracture.) If you cut a foot-rule into four equal parts, each part will be $\frac{1}{4}$ of a foot long. If the foot is divided into 12 equal parts, each part is 1 in long, so that we say that

$$1 \text{ in} = \tfrac{1}{12} \text{ ft}$$

Example:



This diagram of a ruler shows that 3 in is both $\frac{3}{12}$ ft and $\frac{1}{4}$ ft. In other words,

$$\frac{1}{4} = \frac{3}{12}$$

Notice that $\dfrac{3}{12} = \dfrac{1 \times 3}{4 \times 3}$, that is, the fraction $\frac{1}{4}$ is unaltered in size if we multiply both the top (the *numerator*) and the bottom (the *denominator*) by 3. Similarly, if we divide both the numerator and the denominator of the fraction $\frac{3}{12}$ by 3, we get $\frac{1}{4}$, so this also leaves the size of the fraction unaltered.

Example:

We see the same thing again if we imagine a cake shared between 10 children. They will get $\frac{1}{10}$ of the cake, as shown on left.

If, however, after the cake had been cut into 10 slices, it turned out that only 5 children had to share it, they would each get two slices, that is, $\frac{2}{10}$ of the cake, as shown on right.

We see that if we take the fraction $\frac{2}{10}$ and divide both the numerator and the denominator by 2, we get the fraction $\frac{1}{5}$, which is the same size as $\frac{2}{10}$.

We often use this device to make the numbers in a fraction simpler, and the process is called *cancelling*. Thus, to simplify the fraction

$$\frac{14}{21}$$

we divide both the numerator and the denominator by 7, and reduce the fraction to

$$\frac{2}{3}$$

In other words,

$$\frac{14}{21} = \frac{2}{3}$$

Sometimes we find that it is possible to cancel more than once:

$$\frac{18}{42} = \frac{9}{21} \text{ (cancelling 2)}$$
$$= \frac{3}{7} \text{ (cancelling 3)}$$

EXERCISE 1A

Simplify the following fractions by cancelling:

1. $\frac{4}{6}$	**3.** $\frac{2}{4}$	**5.** $\frac{12}{15}$	**7.** $\frac{24}{30}$	**9.** $\frac{42}{63}$
2. $\frac{8}{10}$	**4.** $\frac{3}{9}$	**6.** $\frac{15}{25}$	**8.** $\frac{77}{121}$	**10.** $\frac{30}{135}$

Factors and Prime Numbers

Any number which divides exactly into another number is called a *factor* of the second number. For instance, 6 divides exactly into 42, so 6 is a factor of 42.

Now some numbers have no factors. 7 is an example: there is no number which divides exactly into 7 (except of course 1, and 7 itself, which hardly count).

A number which has no factors (except 1 and itself) is called a *prime number*. You will soon be able to check that 13 and 31 are also prime numbers.

EXERCISE 1B

1. Is 2 a factor of 10?
2. Is 3 a factor of 10?
3. Is 5 a factor of 10?
4. Write down all the numbers which are factors of 12.
5. Is 15 a prime number?
6. Is 17 a prime number?
7. Is 61 a prime number?

Prime Factors and "Powers"

The number 42 can be written as 6×7, or as $2 \times 3 \times 7$. If we write it in either of these ways, we are said to have *expressed* 42 *in factors*. You will see that there are two ways of expressing 42 in factors. (6×7 and 7×6 are not counted as different, nor do we count $3 \times 7 \times 2$ as different from $2 \times 3 \times 7$.)

Of these two ways, $2 \times 3 \times 7$ consists only of prime numbers, whereas the other way uses the number 6, which is not a prime number. When there are several ways of expressing a number in factors, the way in which prime numbers only are used is called expressing the number in *prime factors*. As a further example, the number 12 can be written as

$$\begin{aligned} &2 \times 6 \\ \text{or } &3 \times 4 \\ \text{or } &2 \times 2 \times 3. \end{aligned}$$

Of these, only the last is stated entirely in prime factors.

You will see that the prime factors of 12 contain the number 2 twice. Some numbers give us whole rows of the same number; for example:

$$64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

To shorten the labour of writing all these numbers down, a French mathematician called Descartes thought of the idea of expressing them like this:

$$2^6$$

(A great saving in time and space.) The number 6 is written at the top right-hand corner of the number 2, and is taken to mean six 2's, all in a row, multiplied together.

Notice that it does *not* mean $6 \times 2\ (= 12)$. The number 6 here has several names: it is called a *power*, or an *index*, or an *exponent*. When we want to say it in words we use the phrase

2 to the power of 6

However, in the case of the powers 2 and 3, we avoid the phrase above and use the words *squared* and *cubed*, thus:

3^2 is called "three squared" and means $3 \times 3\ (= 9)$
3^3 is called "three cubed" and means $3 \times 3 \times 3\ (= 27)$
2^4 is called "two to the power four" and means $2 \times 2 \times 2 \times 2\ (= 16)$

By using this method of writing numbers we can express, for example, 72 as

$$\begin{aligned} &2 \times 2 \times 2 \times 3 \times 3 \\ \text{or } &2^3 \times 3^2 \end{aligned}$$

EXERCISE 1C

1. Evaluate (work out): (a) 2^3; (b) 3^2; (c) 5^2; (d) 4^2; (e) 2^5; (f) $2^3 \times 3$; (g) $3^3 \times 2^2$; (h) $2^3 \times 3 \times 5^2$.

2. Write down *all* the ways of expressing 18 in factors; underline the way which uses only prime factors; write your answer using powers.

3. Find the prime factors of 24, writing your answer using exponents.

4. What index attached to the number 3 will give us the answer 81?

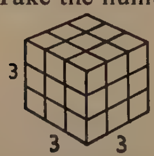
Finding the Prime Factors

To express a large number in prime factors can be too complicated to do by guesswork. A more routine method is to perform a series of successive divisions, as follows:

Example:

Take the number 60: 2 is a factor of 60, thus $60 = 2 \times 30$
2 is a factor of 30, thus $60 = 2 \times 2 \times 15$
3 is a factor of 15, thus $60 = 2 \times 2 \times 3 \times 5$
5 is a prime number, so the calculation is over:

$$60 = 2^2 \times 3 \times 5$$



For compactness, we can set out the work like this:

To find the prime factors of 1760:

$$\begin{array}{r|r} 2 & 1760 \\ \hline 2 & 880 \\ \hline 2 & 440 \\ \hline 2 & 220 \\ \hline 2 & 110 \\ \hline 5 & 55 \\ \hline & 11 \end{array}$$

Answer: $1760 = 2^5 \times 5 \times 11$

To be systematic, we start dividing by 2 and continue as long as possible. Next we try 3 (which did not appear in this example), then 5 (the next prime number). If necessary, we would then try the *next* prime number, 7, and so on. When the quotient is itself a prime number, we stop, and collect up all the factors for the answer.

EXERCISE 1D

Express the following in prime factors:

1. 840. **2.** 364. **3.** 153. **4.** 1326. **5.** 864.

Use of Prime Factors for Cancelling

Expressing numbers in prime factors helps us in cancelling fractions with large numbers in them. For example, take the fraction

$$\frac{552}{828}$$

By putting the numerator and denominator into prime factors, we write the fraction as

$$\frac{2^3 \times 3 \times 23}{2^2 \times 3^2 \times 23}$$

$$\text{or} \quad \frac{2 \times 2 \times 2 \times 3 \times 23}{2 \times 2 \times 3 \times 3 \times 23}$$

and by cancelling two pairs of 2s, a pair of 3s and a pair of 23s, we reduce the fraction to $\frac{2}{3}$. Notice how the 2^3 in the numerator and the 2^2 in the denominator leave a single 2 at the top. In the same way

$$\frac{2^7}{2^3} = \frac{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2}$$

and the three 2s at the bottom will cancel with three of the 2s at the top, leaving four 2s as the answer, that is

$$\frac{2^7}{2^3} = 2^4 = 16$$

The final index, 4, is the *difference* between the two original indices, 7 and 3.

$$\text{Similarly } \frac{2^2}{2^5} = \frac{2 \times 2}{2 \times 2 \times 2 \times 2 \times 2} = \frac{1}{2^3}$$

Here the final index at the *bottom* is the difference of 5 and 2, because the larger index was in the denominator.

EXERCISE 1E

Using your answers to Exercise 1D simplify:

1. $\frac{364}{840}$ **2.** $\frac{153}{864}$ **3.** $\frac{864}{1326}$ **4.** $\frac{840}{864}$

The Highest Common Factor

When we discussed how to cancel the fraction

$$\frac{552}{828}$$

we first of all wrote it as $\dfrac{2^3 \times 3 \times 23}{2^2 \times 3^2 \times 23}$

and then cancelled $2^2 \times 3 \times 23$ ($= 12 \times 23 = 276$). This means that 276 is a factor of both 552 and 828; in fact:

$$552 = 276 \times 2$$
$$\text{and } 828 = 276 \times 3$$

Since there is clearly no larger number that is a factor of both 552 and 828, we say that 276 is the *highest common factor* (H.C.F. for short) of these two numbers. To find the H.C.F. of two (or more) numbers, we must first express them in prime factors, and then select all the prime factors that occur in both numbers, thus:

To find the H.C.F. of 364 and 504:

$$364 = 2^2 \times 7 \times 13$$
$$504 = 2^3 \times 3^2 \times 7$$

The common prime factors of these two numbers are

$$2^2 \times 7$$

so the H.C.F. is 28

EXERCISE 1F

Find the H.C.F. of:

1. 12 and 18.
2. 68 and 306.
3. 45 and 300.
4. 528 and 572.
5. 500, 375 and 825.

6. What is the greatest width of carpeting which will serve to fit exactly two rooms, one 9 ft wide and the other 11 ft 3 in wide?

Multiplying Fractions

If a yard of ribbon cost 6p, then four yards will cost

$$4 \times 6\text{p} = 24\text{p}$$

Obviously, a situation like this is dealt with by multiplication, so it would seem sensible to say that $\frac{2}{3}$ of a yard of ribbon costs $\frac{2}{3} \times 6$p. But how are we to multiply together a fraction and a whole number? Now we know that if a yard costs 6p, then a foot will cost 2p and 2 ft will cost 4p. Since $\frac{2}{3}$ yd $= 2$ ft, the calculation

$$\frac{2}{3} \times 6\text{p}$$

ought to give us the answer 4p. To make sense, we should have

$$\frac{2}{3} \times 6 = 4$$

Now notice that $\dfrac{2 \times 6}{3} = \dfrac{12}{3} = 4$

which leads us to suggest that the sensible way to multiply a fraction and a whole number is to multiply the *top* of the fraction by the whole number

and to leave the bottom as it is. Because this rule gives us sensible answers, it is in fact the rule that mathematicians have adopted. Thus:

$$\frac{3}{8} \times 16 = \frac{3 \times 16}{8} = \frac{48}{8} = 6 \text{ (by the rule)}$$

and this makes sense in that if, say, a gallon of spirit costs 16p, then a pint costs 2p, and three pints, which is $\frac{3}{8}$ of a gallon, costs 6p.

The same kind of situation shows us how to multiply two fractions. If a boy is paid 20p for each hour he works in a neighbour's garden, and he works for $1\frac{1}{2}$ hours, he should receive 30p; we might write

$$1\tfrac{1}{2} \times 20\text{p} = 30\text{p}$$

Suppose we look at the problem another way: he works for three half-hours, i.e. $\frac{3}{2}$h, he is paid at £$\frac{1}{5}$ for each hour, and he receives £$\frac{3}{10}$. We should be able to write

$$\frac{3}{2} \times \frac{1}{5} = \frac{3}{10}$$

Now notice that $\dfrac{3 \times 1}{2 \times 5} = \dfrac{3}{10}$, which suggests that the sensible way to multiply two fractions is to multiply the tops and to multiply the bottoms. Once again, this is the sensible rule that mathematicians have adopted.

Proper and Improper Fractions

In the last example, we did not write

$$1\tfrac{1}{2} \times \tfrac{1}{5}$$

but

$$\tfrac{3}{2} \times \tfrac{1}{5}$$

A fraction like $\frac{3}{2}$, in which the top is larger than the bottom, is called an *improper fraction*. A *proper* fraction is one in which the top is smaller than the bottom, such as $\frac{1}{5}$. An improper fraction can always be reduced by division either to a whole number ($\frac{12}{3} = 4$) or to a *mixed number* ($\frac{3}{2} = 1\frac{1}{2}$). A mixed number can be turned into an improper fraction like this:

$$\begin{aligned} 2\tfrac{3}{8} &= 2 + \tfrac{3}{8} \\ &= \tfrac{16}{8} + \tfrac{3}{8} \\ &= \tfrac{19}{8} \end{aligned}$$

If we have a multiplication involving a mixed number, we cannot use the rule discussed above, so we turn the mixed number into an improper fraction

first. Thus to find the cost of $1\frac{2}{3}$ yd at $22\frac{1}{2}$p per yard, we must write

$$\text{Cost is } 1\tfrac{2}{3} \times 22\tfrac{1}{2}\text{p}$$

$$= \frac{5}{3} \times \frac{45}{2}\text{ p} \qquad\qquad \frac{5}{\cancel{3}_{1}} \times \frac{\cancel{45}^{15}}{2}$$

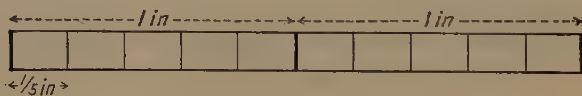
$$= \frac{5 \times 15}{1 \times 2}\text{ p}$$

$$= \frac{75}{2}\text{ p} = 37\tfrac{1}{2}\text{p}$$

(Notice how the fractions were cancelled *before* the tops and bottoms were multiplied; this can save a lot of heavy arithmetic, but should be done carefully and neatly.)

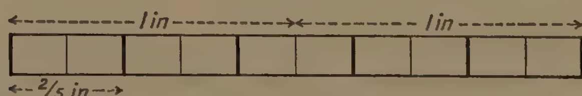
Division by Fractions

To investigate the division of one fraction by another, let us first start with the simple problem of dividing 2 by $\frac{1}{5}$: this can be illustrated by the problem: how many times will $\frac{1}{5}$ inch fit into 2 inches.



Since $\frac{1}{5}$ inch fits into 1 inch 5 times, it will fit into 2 inches 2×5 times, or 10 times. Notice how we have multiplied 2 by the *bottom* of the fraction, 5.

Now let us see how many times $\frac{2}{5}$ in fits into 2 in. Clearly since $\frac{2}{5}$ in is twice as big as $\frac{1}{5}$ in, it will fit into 2 in only half as many times as $\frac{1}{5}$ in did: that is, $\frac{10}{2}$ times, or 5 times, as can be seen in this next diagram:



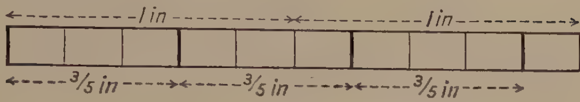
Notice that the answer, 5, is derived by multiplying 2 by the bottom of the fraction, 5, and then dividing by the top, 2. In other words,

$$2 \times \tfrac{5}{2} = 5$$

If we use the same process to find how many times $\frac{3}{5}$ in fits into 2 in we may expect to get the answer by multiplying 2 by the bottom of the fraction, 5, and then dividing by the top, 3. This will give

$$2 \times \tfrac{5}{3} = \tfrac{10}{3} = 3\tfrac{1}{3}$$

The diagram shows that this is the case.



Our investigation leads us to the general rule: to divide by a fraction, multiply by the bottom and divide by the top, or more simply,

turn it upside down and multiply.

We were only concerned just now with the division of a whole number by a fraction. If we apply the rule to, say,

$$\tfrac{3}{4} \div \tfrac{7}{9}$$

we shall get $\tfrac{3}{4} \times \tfrac{9}{7} = \tfrac{27}{28}$

To check this, let us write the problem as

$$\frac{\tfrac{3}{4}}{\tfrac{7}{9}}$$

$$= \frac{\tfrac{3}{4} \times 36}{\tfrac{7}{9} \times 36}$$

$$= \tfrac{27}{28}$$

which gives us the same answer, so we may safely use the rule in all circumstances.

EXERCISE 1G

Evaluate:

1. $\tfrac{3}{4} \times \tfrac{5}{9}$
2. $\tfrac{2}{5} \times \tfrac{3}{8}$
3. $\tfrac{3}{7} \times \tfrac{14}{15}$
4. $\tfrac{2}{3} \div \tfrac{4}{5}$
5. $\tfrac{11}{14} \div \tfrac{5}{7}$
6. $\tfrac{5}{8} \div 1\tfrac{7}{16}$
7. $2\tfrac{1}{2} \times \tfrac{3}{5}$
8. $3\tfrac{3}{4} \times 1\tfrac{3}{5}$
9. $7\tfrac{1}{8} \div 4\tfrac{3}{4}$
10. $2\tfrac{1}{3} \times \tfrac{3}{7}$

CHAPTER 2

MENSURATION I

THE STUDY of Arithmetic covers two main items: number and measurement. Sometimes we concentrate on one, sometimes on the other, but the two are so closely connected that usually we do not bother to distinguish between them. For example the remark

$$3 \times 2 = 6$$

is a number fact. We use this fact in the simple problem: If the water in a bucket just fills three 2-pint bottles, how much water was there in the bucket? The answer is clearly

$$3 \times 2 \text{ pt} = 6 \text{ pt}$$

We used our knowledge of the multiplication tables (which are number facts) to solve the problem in measurement. But there is one important difference in the two sorts of answer: measurements are made in some sort of units (in the problem above, pints), so the answer to a measurement problem *must* be expressed in some units. Unless the unit is given, the answer is *wrong*.

The word *mensuration* which heads this chapter means the use of calculation to solve problems in measurement. Since the unit involved is an essential part of the answer, we will start by reviewing the method of changing a measurement from one unit to another. Suppose we want to change 2 ft 4 in to inches. Look at the diagram:

Each foot contains 12 in, so 2 ft contains 2×12 in $= 24$ in.

There is still 4 in left over, which must be added to the 24 in we have so far, so the final answer is:

$$24 \text{ in} + 4 \text{ in} = 28 \text{ in}$$

This problem needed a piece of multiplication, because we were moving from a large unit to a smaller unit (feet to inches). The next one will need a division sum, because we shall be moving from a small unit to a larger unit.

Example:

Express 30 pints in gallons and pints.

Since 1 gal = 8 pt, we can set out our 30 pt and group them in eights, thus:

How many gallons in 30 *pints?*

There are 3 groups of 8 pt, which make 3 gal, and there are 6 pt over. This corresponds to the division sum

$$\begin{array}{r} 3 \\ 8\overline{)30} \\ \underline{24} \\ 6 \end{array} \text{ remainder}$$

so we may give the answer as

$$30 \text{ pt} = 3 \text{ gal } 6 \text{ pt}$$
$$\text{or } 30 \text{ pt} = 3\tfrac{3}{4} \text{ gal}$$

EXERCISE 2A

Express the given quantities in terms of the units indicated:

1. 8 yd 2 ft (ft)
2. £4·60 (p)
3. 3 tons 16 cwt (cwt)
4. 48 in (ft)
5. 26 pt (gal, pt)
6. 128p (£)
7. 40 oz (lb)
8. 250 s (min, s)

Uniformity of Units

If we want to express 4 in as a fraction of a foot, we say that 1 ft = 12 in and so 4 in = $\frac{4}{12}$ ft = $\frac{1}{3}$ ft (cancelling 4). Before doing the sum, we had to express 1 ft in inches. It is important that when we try to express one quantity as a fraction of another, we first ensure that they are in the same unit.

Example:

To express 2 ft 4 in as a fraction of 5 ft 10 in, we must say

$$\begin{aligned} 2 \text{ ft } 4 \text{ in} &= (2 \times 12) \text{ in} + 4 \text{ in} \\ &= 24 \text{ in} + 4 \text{ in} \\ &= 28 \text{ in} \\ \text{and } 5 \text{ ft } 10 \text{ in} &= (5 \times 12) \text{ in} + 10 \text{ in} \\ &= 60 \text{ in} + 10 \text{ in} \\ &= 70 \text{ in} \end{aligned}$$

so the fraction is $\frac{28}{70} = \frac{2}{5}$ (cancelling 14)

EXERCISE 2B

Express the first quantity in each question as a fraction of the second:

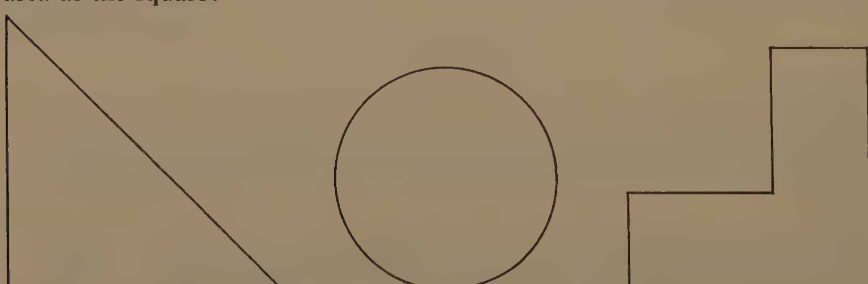
1. 8 in, 2 ft
2. 20 s, 1 min 20 s
3. 8 oz, 1 lb 4 oz
4. 3 pt, 3 gal
5. 40p, £1·40
6. 90 min, 24 h

The Measurement of Areas

The previous exercise was largely concerned with the revision of some of the units of length, money, time, weight and capacity. We now come to consider the measurement of area. One suitable unit for areas is the *square inch.* Here is an area of one square inch:

A 1-*in square.*

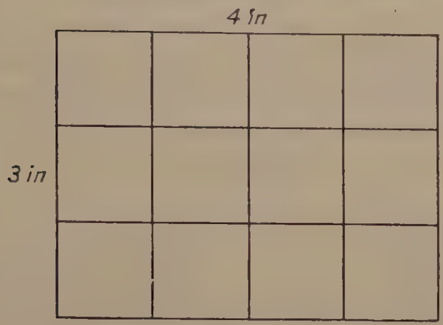
You will see that it is a square whose edges are one inch long. Naturally, the shape need not be a square: we could cut the shape into various pieces and re-arrange them to make all sorts of figures which have the same area—one square inch. Here are some examples of figures which all have the same area as the square:



Three shapes each having an area of 1 *square inch.*

However, the square shape is the easiest to work with, and we express the area of a figure by saying how many of these square inches can be fitted into it.

Look at the next diagram of a rectangle 4 in long and 3 in wide (it is drawn smaller than life to save space).



Twelve of our square inches just fit into the rectangle, in three rows of four (because the diagram is 3 in wide and 4 in long). Similarly, a rectangle 7 in long and 2 in wide will contain two rows, each of seven square inches, giving an area of 14 square inches altogether.

It is easy to see that to find the area of *any* rectangle, we take the number of inches in its length (which gives us how many square inches there are in *one* row) and multiply by the number of inches in its breadth (which tells us how many such rows there are) and this will tell us how many square inches will fit into the rectangle. Put shortly, we can say:

area of rectangle in square inches = length in inches × breadth in inches

We could almost say that, when finding the area of a rectangle,

$$(\text{inches}) \times (\text{inches}) = (\text{square inches})$$

and we saw in Chapter I that we could write

$$8 \times 8 = 8^2.$$

If we follow this line of thought a little further, we can now write

$$(\text{in}) \times (\text{in}) = (\text{in}^2)$$

This kind of thinking has led to the adoption of the symbol

$$\text{in}^2$$

for an area in square inches. Just as we write that the length of the rectangle above is 4 in, so we write that its area is 12 in².

We can use other units of area besides the square inch. Suppose a room is 15 ft long and 12 ft wide; then the floor could be covered by 180 (= 15 × 12) squares whose sides are 1 ft long. We say that the area is 180 square feet, or

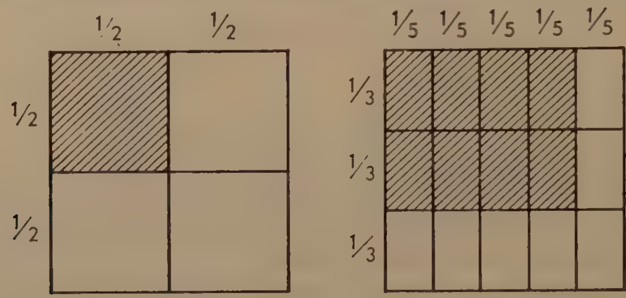
$$180 \text{ ft}^2$$

Another possible unit is the *square yard*. The floor in the example above is 5 yd long and 4 yd wide, so its area is

$$5 \times 4 \text{ yd}^2$$
$$= 20 \text{ yd}^2$$

The important thing to remember when finding the area of a rectangle is that both the length and breadth must be expressed in the same unit, such as inches, feet or yards, and then when you multiply them together, the area will be so many square inches, square feet or square yards.

We have only shown that this rule for finding the area of a rectangle works for rectangles whose sides are measured in whole numbers of units. If, however, we use the rule together with the rule for multiplying fractions given in Chapter 1, we find that it still applies (which is another reason for adopting the rule in Chapter 1). Look at these diagrams:



$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ $\qquad$ $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$

The shaded rectangle in the left-hand diagram has length and breadth each $\frac{1}{2}$ in, so the area rule and the fraction rule give us an area of

$$\tfrac{1}{2} \times \tfrac{1}{2}\ \text{in}^2 = \tfrac{1}{4}\ \text{in}^2$$

and we can see that the area is indeed a quarter of a square inch. In the same way the right-hand diagram shows a rectangle whose area is

$$\tfrac{2}{3} \times \tfrac{4}{5}\ \text{in}^2$$
$$= \tfrac{8}{15}\ \text{in}^2$$

EXERCISE 2C

In each question you are given the length and breadth of a rectangle: express its area in the units indicated.

1. 6 in, 3 in (in²)
2. 8 ft, 5 ft (ft²)
3. 3 yd, 2 yd (yd²)
4. 12 miles, 10 miles (mile²)
5. 2 ft, 7 in (in²)
6. 2 yd, 2 ft (ft²)
7. 2 ft 3 in, 1 ft 8 in (in²)

A Carpet Problem

Suppose a housewife is going to buy some carpet for her sitting-room. She measures the floor carefully and finds that she needs 180 ft² of carpet. Now in the shop this carpet is sold in long rolls which are 3 ft wide. You can buy whatever length you need. What length did this lady need?

Here is a diagram of the length of carpet:

Now the area is found by multiplying the number of feet in length by 3 ft, and this sum must have the answer 180 ft². Three times what number equals 180? To find the answer, we must divide 180 by 3.

$$3\overline{)180}^{\;60}$$

The answer is 60 ft. This is easily checked, as a rectangle 60 ft long and 3 ft wide has an area of 60×3 ft²

$$= 180 \text{ ft}^2$$

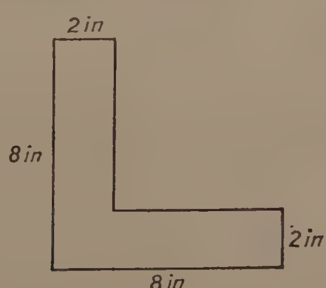
EXERCISE 2D

In the following questions, the area of a rectangle is given, and either the length or the breadth. You are to find whichever measurement is missing:

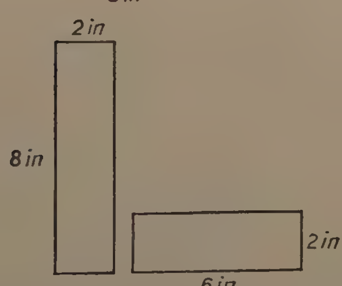
1. 8 in², length 4 in.
2. 30 ft², breadth 5 ft.
3. 324 yd², breadth 12 yd.
4. 252 ft², length 18 ft.
5. 180 in², breadth 1 ft.
6. 2160 in², length 2 yd.

Area Measurement by Sections

Sometimes we can find the area of a more complicated figure than a rectangle by dividing it into two or more pieces which *are* rectangular in shape. For instance, the letter L shown here



can be divided like this



and so is seen to consist of two rectangles, one 8 in long and 2 in wide, and the other 6 in long and 2 in wide.

By working out the area of each rectangle and then adding them together, we can find the area of the whole figure, thus:

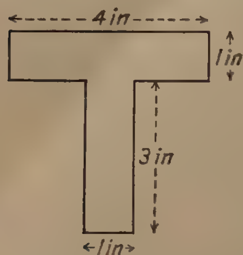
$$\begin{aligned} 8 \text{ in} \times 2 \text{ in} &= 16 \text{ in}^2 \\ 6 \text{ in} \times 2 \text{ in} &= \underline{12 \text{ in}^2} \\ & \underline{28 \text{ in}^2} \end{aligned}$$

Answer: 28 in²

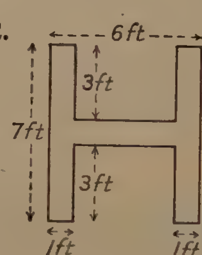
EXERCISE 2E

Find the areas of the following shapes by the method just shown:

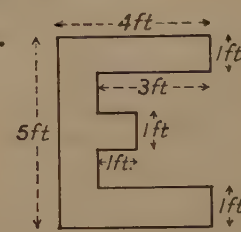
1.



2.

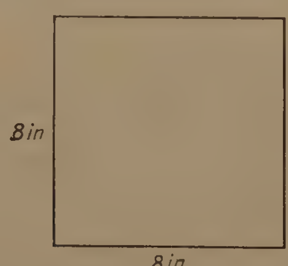


3.

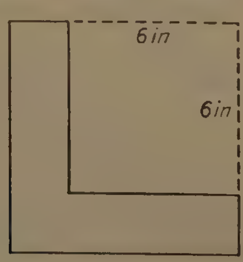


Finding Area by Subtraction

Let us go back for another look at the L-shaped figure in the last example. If we were going to make it out of paper or card, we should probably start with a square piece of side 8 in, like this.



and then cut out at one corner a square of side 6 in, thus:



We started with a piece of area 8 in $\times$ 8 in = 64 in², and took away a piece of area 6 in $\times$ 6 in = 36 in². The area left, which is the area we want, is therefore

$$64 - 36 \text{ in}^2 = 28 \text{ in}^2, \text{ as before.}$$

This method of finding an area by subtraction can be very useful, especially when we are trying to find the area of a border round a rectangle. Suppose we have an enlargement of a holiday photograph which measures 9 in long and 7 in wide, and we mount it on a piece of card so that there is a border left 1 in wide all round, like this:

First of all, what are the dimensions of the card? Since it is 1 in longer *at each side* of the photograph, it must be 2 in longer altogether, so it must be 11 in long. In the same way, it must be 9 in wide. Thus its area is

$$11 \text{ in} \times 9 \text{ in} = 99 \text{ in}^2$$

Now the area of the photograph is

$$9 \text{ in} \times 7 \text{ in} = 63 \text{ in}^2$$

so the area of the border round the edge of the photograph is

$$99 - 63 \text{ in}^2 = 36 \text{ in}^2.$$

EXERCISE 2F

1. A man is going to paint one wall of a room. The wall is 12 ft long and 9 ft high, and contains a doorway 3 ft wide and 7 ft high. What is the area of the wall?

2. A garden is 60 ft long and 24 ft wide, and a concrete path 3 ft wide runs all round within it. What is the area of the path?

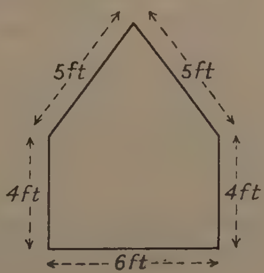
3. Find the area of the letter H in Exercise 2E by the subtraction method.

Perimeters

If you were to walk all round the edge of a square field whose sides were 150 yd long, how far would you walk? Obviously 4 $\times$ 150 yd, which comes to 600 yd. This distance is called the *perimeter* of the field.

It is easy to find the perimeter of any figure just by adding up the lengths of all its sides. The figure need not be square, and can have any number of sides. For instance, the perimeter of this figure is

$$(4 + 6 + 4 + 5 + 5) \text{ ft} = 24 \text{ ft.}$$

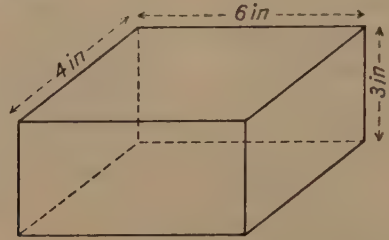


EXERCISE 2G

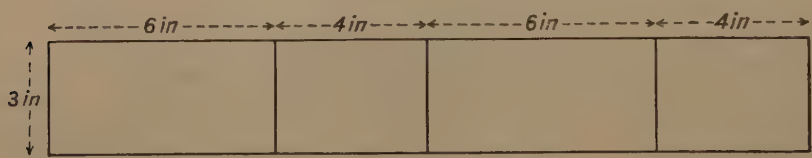
Find the perimeters of the three figures in Exercise 2E.

The Area of a Box

Now suppose we are trying to find the total area of the four *sides* of a cardboard box like this:



Since we are not interested in the bottom, let us cut it away, and then we will flatten out the four sides to make one long rectangle, like this:



The *breadth* of this rectangle is the same as the *height* of the original box, and the *length* is the same as the *perimeter* of the box, since it represents the distance round the bottom, which we cut away. Thus the sides of the box can make a rectangle 20 in long and 3 in broad, so the area of the four sides comes to 3 in $\times$ 20 in $=$ 60 in^2.

The same method can be applied to, say, the four walls of a room. Obviously we cannot flatten them out as we did with the cardboard box, but we can *imagine* it being done, and so by calculating

$$(\text{perimeter}) \times (\text{height})$$

we can find the area of the walls of the room.

Of course, if we want to find the total area of the *whole* room, including the floor and the ceiling, we must first find the area of the walls in the manner described, then find by a separate calculation the area of the floor and of the ceiling, and finally add all our results together.

EXERCISE 2H

Use the formula (perimeter) $\times$ (height) to help find the following areas:

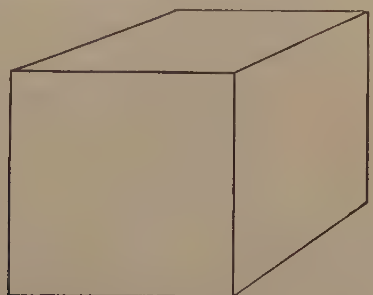
1. The walls of a room 18 ft long, 14 ft broad and 9 ft high.

2. The *total* area (including the base) of a water-tank 4 ft high whose base is a square of side 5 ft (the tank has no top).

3. The total area of a *closed* box 10 in long, 4 in wide and 3 in high.

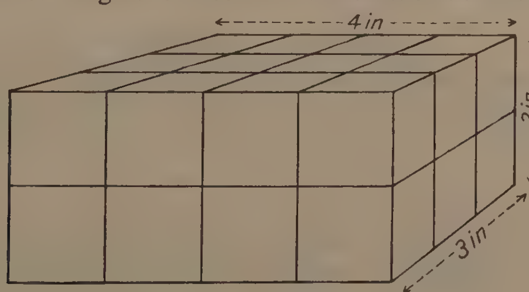
Cubic Measurement

You have seen that to measure area we use as a unit a square of some suitable size, such as a square foot. In the same way, to measure *volume*, we use as a unit a cube of suitable size, such as the cube shown here. The volume occupied by this cube is called *one cubic inch.*



One cubic inch.

To find the volume of a box we must calculate how many of these cubic inches will fit into it. If, for example, we have a box 4 in long, 3 in wide and 2 in high, we must imagine it filled with these little cubic inches, like this:



Block of 24 1*-inch cubes.*

You will see that there are two layers of cubic inches, one on top of the other; this is because the box is 2 in high. Since the box is 4 in long and 3 in wide, each layer consists of 3 rows, each containing 4 cubes. That is, there are $4 \times 3 = 12$ cubes in each layer, and since there are two layers, the box contains $12 \times 2 = 24$ cubic inches altogether. The symbol for cubic inches is in^3, so we say that the volume of this box is 24 in^3.

Of course, we can do the same sort of thing for a box of any size. We multiply the length by the breadth to find the number of cubes in the bottom

layer, and then multiply that answer by the number of layers (that is, the height). Put more shortly, we can say

$$\text{Volume of box} = (\text{length}) \times (\text{breadth}) \times (\text{height})$$

provided that we remember to work entirely in one unit, and not mix up yards and feet, say, in one problem. If a box measures 2 ft by 1 ft by 5 in, its volume is

$$\begin{aligned} & 24 \times 12 \times 5 \text{ in}^3 \\ = & \ 288 \times 5 \text{ in}^3 \\ = & \ 1440 \text{ in}^3. \end{aligned}$$

EXERCISE 2J

1. Find the volume of a box measuring 6 ft by 4 ft by 3 ft.

2. Find the volume of air in a room 3 yd high if the floor measures 5 yd by 4 yd.

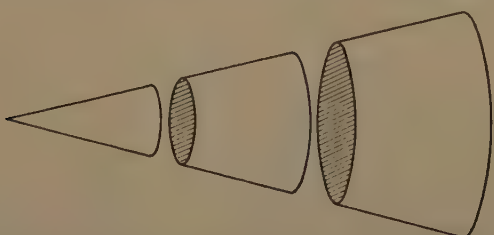
3. Find the volume of a water tank whose base is a square of side 2 yd if the tank is 4 ft high.

4. A biscuit tin measures 10 in by 8 in by 4 in. Find its volume. How many such tins can be packed into a carton 2 ft 6 in long, 1 ft 4 in wide and 1 ft high?

5. The length and breadth of a box are 6 in and 5 in. The volume of the box is 60 in³. Find its height.

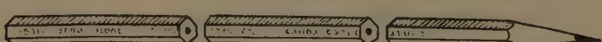
Cross-section Dimensions

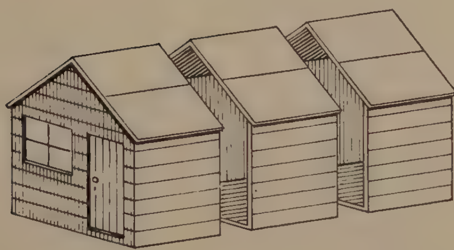
If we cut a solid object at right angles to its length, we get a flat surface which is called a *cross-section* of the object. If we cut it in different places, the cross-sections will not necessarily be the same size or shape.



Cone cut in three to show varying cross-sections.

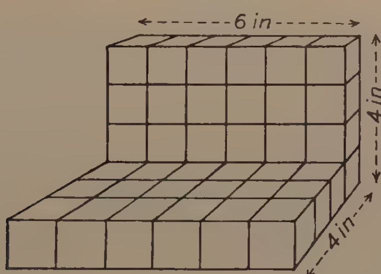
If, however, we get exactly the same cross-section no matter where we cut it, the object is said to have *uniform cross-section*. A common type of pencil has a uniform cross-section (except at the sharpened end).





So has a simple garden shed, or a tin can.

To see how to find the volume of such an object, consider one of this pair of book-ends.



If we imagine it made up of cubic inches, like this, you will see that it is composed of 6 L-shaped pieces stuck together. By counting, we find that there are 7 cubic inches in each piece, and so the total volume is

$$6 \times 7 \text{ in}^3 = 42 \text{ in}^3$$

Now, although we are talking about cubic inches, you will see that the 7 cubic inches that form one piece correspond to the 7 *square* inches which make up the *area of the cross-section*, while the 6 pieces correspond to the fact that the book-end is 6 in *long*.

Thus the volume of the solid is

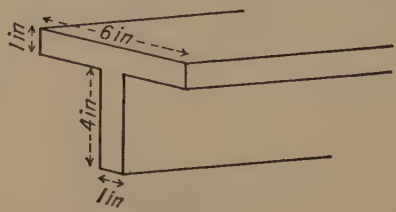
$$(\text{area of cross-section}) \times (\text{length})$$

We can use this method to find the volume of any object which has a *uniform* cross-section.

EXERCISE 2K

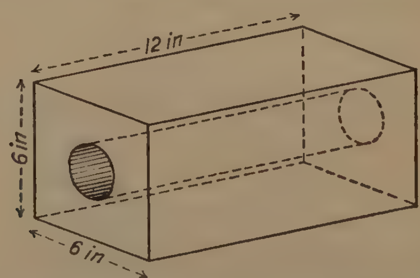
1. A can of baked beans has a circular base of area 8 in² and it is 5 in high. Find its volume.

2. The cross-section dimensions of a T-girder 10 ft long, are shown here. Find its volume.



Section of T-girder.

3. A block of 6 × 6 in wood, 12 in long, has a circular hole bored through it, lengthwise. The cross-section area of the hole is 12 in². Find the volume of the wood remaining.



12-*in length of* 6 × 6 *in wood with hole* (*of* 12 *in*² *cross-section*) *drilled through it.*

CHAPTER 3

PROPORTION, SPEED AND RATES

Direct Proportion

IF WE go into a shop, see that sweets are 7p per quarter, and buy 2 lb, we need to do a very simple proportion sum to know what to pay. The weight of the sweets and their cost increase in the same way. If we double the weight of sweets, we pay twice the amount, if we buy four times the weight, we pay four times the money.

If a train travels at a steady speed and covers 40 miles in 1 hour, it will cover 80 miles in 2 hours. If the time is doubled, so is the distance.

When quantities are connected like this, they are in direct proportion to each other.

EXERCISE 3A

1. What is the cost of 4 lb of butter at 16p per pound?

2. If a plane flies at an average speed of 250 miles per hour, how far will it travel in 3 hours?

3. How much will 15 lb of apples cost if 1 lb costs 9p?

4. If 6 ft of planking weighs 18 lb, what will 1 ft of planking weigh?

5. What would one text-book cost if 30 text-books cost £7·50?

6. If a watch gains at the constant rate of 2 seconds per hour, what will it gain in a day?

It sometimes helps in problems like these to find out what one unit would cost. Because the second line deals with *one* gallon (or one yard, or one egg, depending on the sum), this is called the *unitary* method.

Example:

What is the cost of 12 gal of paraffin if 18 gal cost £1·80?

18 gal cost 180p

$\therefore$ 1 gal costs $\frac{180}{18}$p (since $\frac{1}{18}$ of the quantity will cost $\frac{1}{18}$ of the price)

$= 10$p

$\therefore$ 12 gal cost 12×10p (since 12 gal will cost 12 times as much as 1 gal)

$= 120$p or £1·20

Example:

If I can buy 5 ball-point pens for 50p, how many pens can I buy for 90p?

1. Ask yourself first "What have I to find"? In this case it is the number of pens.
2. Write the first line so that the quantity you have to find is at the *right-hand* side.
3. Remember to put the units (in this case pens) in your answer.

50p buys 5 pens

$\therefore$ 1p buys $\frac{5}{50}$ pens

$\therefore$ 90p buys $\frac{5 \times 90}{50}$ pens $= \frac{\overset{1}{\cancel{5}} \times 90}{\underset{1}{\cancel{50}}}$

$= 9$ pens

Example:

If a car will travel 132 miles on 4 gal of petrol, how far will it travel on $7\frac{1}{3}$ gal if it uses petrol at the same rate?

(We have to find "how far.")

On 4 gal the car will travel 132 miles.

On 1 gal the car will travel $\frac{132}{4}$ miles.

On $7\frac{1}{3}$ gal the car will travel $\frac{132}{4} \times \frac{22}{3}$ miles

$= 242$ miles

$$\frac{\overset{11}{\overset{\cancel{33}}{\cancel{132}}}}{\underset{1}{\cancel{4}}} \times \frac{22}{\underset{1}{\cancel{3}}}$$

EXERCISE 3B

1. Find the cost of 12 yd² of carpet if 5 yd² cost £14·25.

2. If a walker covers 15 miles in 5 hours (walking at a steady rate) what distance will he cover in 1 hour? In 4 hours?

3. Three tons of coke cost £57. How much will 8 tons cost? If a boiler uses 2 tons a month, what will be the yearly bill for coke?

4. An author calculates that he writes an average of 600 words in 125 minutes. How many would he write in 10 minutes at the same rate?

5. A watch gains 15 seconds in 24 hours. How much would it gain in 8 hours at the same rate? In the month of June?

6. A goods train travels at 30 miles per hour. How many minutes would it take to travel 16 miles?

7. If a car travelled 84 miles in 210 minutes, how long would it take to travel 30 miles?

8. If 6 lb of tea cost £2·70, find the cost of 11 lb.

9. A hotel charges £18·20 a week for a certain room. What should a visitor who has only stayed 3 days pay for the room?

10. A boarding school of 85 boys uses 30 lb of sugar in a week. Next term there are to be 102 boys. If they use sugar at the same rate, how much will have to be ordered?

11. If 14 French francs = £1, how many pence is 70 francs worth?

12. A parquet floor 21 ft × 12 ft costs £70 to lay. How much per square yard is this?

Inverse Proportion

Some quantities are connected in a different way:

Example:

A train travelling between Milan and Rome does the journey in 10 hours at an average speed of 39 mile/h. If the average speed is reduced by bad weather to 30 mile/h, how long will the train take?

If the train is slower (the speed *decreases*) it will take longer (the time *increases*), and quantities connected in this way are said to be in *inverse* proportion.

The sum is set out in the same way:

At a speed of 39 mile/h the train takes 10 hours.

∴ At a speed of 1 mile/h the train takes 10 × 39 hours.

$$\therefore \text{ At a speed of 30 mile/h the train takes } \frac{10 \times 39}{30} \text{ hours} = \frac{\cancel{10}^{1} \times \cancel{39}^{13}}{\cancel{30}_{1}}$$

$$= 13 \text{ hours}$$

Notice that at 1 *mile/h the train takes not* $\frac{10}{39}$ *hours, but* 10 × 39 *hours.*

Example:

Six men are engaged to dig the foundations of a house and are expected to take 10 days. Only 5 turn up for the job. How long should they take, if we assume they work at the same pace?

6 men can do the work in 10 days.

∴ 1 man would take 10 × 6 days

$$\therefore \text{ 5 men will take } \frac{10 \times 6}{5} \text{ days}$$

$$= 12 \text{ days}$$

EXERCISE 3C

1. If a plane averages 250 mile/h, it covers the distance from Le Touquet to Nice in 3 hours. How long will it take if a tail wind increases the average speed to 300 mile/h?

2. If I plant lettuces 8 in apart I can get 96 in a row. How many will I be able to plant in the same row if I set them 12 in apart?

3. Three men paint a house in 15 hours. How long will 5 men take, working at the same speed?

4. William has some book tokens for Christmas. He had enough tokens to buy 6 books at 40p each. How many books at 60p each could he have bought?

5. Two girls were knitting with similar needles and pattern. One knitted 10 stitches to the inch and the completed panel was 12 in wide. The other knitted 12 stitches to the inch. How wide was her completed panel?

6. It required 36 strips of wallpaper 21 in wide to cover a wall. If French wallpaper, width 18 in, is used, how many strips will be required?

7. A herd of 15 cows will graze a field in 13 days. How long would it take a herd of 39 cows to graze the same field at the same rate?

8. A machine gun has two rates of fire. At the fast rate, 250 rounds per minute, the magazine is empty in 12 seconds. How long will it fire at the slow rate of only 1 every 2 seconds?

9. From Leeds to London by tandem, at an average speed of 19 mile/h takes 10 hours. How long would it take by helicopter at 114 mile/h? By train at 57 mile/h?

10. The Commander of a besieged garrison of 75 soldiers calculates that he has enough food for 14 days, but almost immediately 5 of his men are shot. How long can the survivors now last before their food is gone?

Non-proportional Relationships

Many quantities are not connected proportionally at all. If a centre-forward in a game of soccer scores 1 goal in 10 minutes, it is not at all certain that he will score 9 goals in 90 minutes. Nor can a golfer who takes only 4 strokes for the first hole be sure that he will complete 18 holes in 72 strokes; he might take less or more. And although a pound of potatoes might cost 2p, the greengrocer might be willing to make a reduction in price to someone who bought 56 lb at one time.

There are, too, other kinds of relationship between quantities. A table-top 2 ft square needs 4 ft^2 of wood. If the length of the sides of the table is doubled the area of wood required is 16 ft^2, *not* 8 ft^2.

In the following exercise there are examples of both direct and inverse proportion, together with some which have no proportional connection.

EXERCISE 3D

1. Jean saved 70p in 49 weeks. How long did it take her to save 40p at the same rate?

2. A tramp steamer takes 15 days for a journey at an average speed of 10 knots. How long would the same journey take if, owing to an engine fault, she only averaged 6 knots?

3. Christopher has 10 sums for his homework. If he does the first 3 in 5 minutes, how long does his homework take?

4. A sprinter runs 100 yd in 10 seconds. How long will he take for 440 yd?

5. My old car, which did 22 miles per gallon, cost me £57 in petrol. What should my new car, which does 38 miles per gallon, cost me if I cover the same mileage, and petrol is the same price?

6. The opening batsmen in a cricket team scored 55 each before they were out. What was the final score after the whole team had batted?

7. If a train travels at a speed of 45 mile/h, how many feet does it travel in 1 second? In n seconds?

8. A gardener digs 3 yd of his border in 40 minutes. How long will he take to dig it all, if it is 13 yd long?

9. A swimming bath was being refilled after its annual cleaning. Last year the bath was filled in 24 hours with the water being pumped in at 100 gal per hour. This year a new machine has been fitted which pumps at 180 gal per hour. If the bath attendant turns on the pump at 7 a.m., at what time will he have to turn it off, if the bath is not to overflow?

10. A dining hall holds 420 boys, each on seats of 15 in wide. New chairs are ordered, 18 in wide. How many boys can now be seated?

11. Corporal Brown is ordered to pace out the parade ground. He marches in steps of 30 in and takes 315 paces. Sergeant Smith decides to check on this by pacing it himself. He has a pace of 35 in. How many steps will he take?

12. If it takes 5 minutes for 5 cats to catch 5 mice, how long will it take for 20 cats to catch 20 mice?

13. A window cleaner, with 3 assistants, can clean all the windows of a factory in 9 hours. In order to avoid having to come back for two days running (since his men only work an 8-hour day) he employs another man. Will he now be able to finish the work in one day if they all work at the same rate? How long will they in fact take?

14. An Atlantic liner makes the crossing from New York to Southampton in 4 days, averaging 30 knots. If bad weather reduced the average speed to 25 knots, how many hours behind schedule will she be on arrival?

15. A gas-holder can be filled in 48 hours if 2 supply pipes are turned on. How long will it take to fill if (a) 3 are used? (b) x pipes are used?

Rates of Change and Speed

A bricklayer earns £1·80 in 3 hours (or at the rate of 60*p per hour*).

A boy walks 12 miles in 4 hours (or at the rate of 3 *miles per hour*).

I buy 3 theatre tickets for £2·25 (or at the rate of 75*p per ticket*).

These examples show that the word "rate" is used to show the connection between two quantities, *money* earned in an *hour*, *miles* walked in an *hour*, number of *shillings* for each *ticket*.

Of course, we could say, in the first example, that the bricklayer earns money at the *rate* of 180p in 3 hours. The connection between the two quantities is still there, but to make a rate useful for calculating other quantities (e.g., how much he earns in 5 hours) it is usual to express it in as simple a way as possible—the amount he earns in *one* hour, the cost of *one* ticket, and so on.

It is particularly useful to express speed as a rate, and by doing so we can compare one speed with another. If we know destroyer A can cover 600 nautical miles in 20 hours, and destroyer B 558 nautical miles in 18 hours, it is difficult to compare their speeds directly, but if we change both speeds to a rate *per hour*:

Destroyer A

In 20 hours it travels 600 nautical miles

$\therefore$ In 1 hour it travels $\frac{600}{20}$ nautical miles

= a speed of 30 knots (nautical miles per hour)

Destroyer B

In 18 hours it travels 558 nautical miles

$\therefore$ In 1 hour it travels $\frac{558}{18}$ nautical miles

= a speed of 31 knots

we can see immediately that Destroyer B is the faster.

If we use easy examples,

30 miles in 2 hours = 15 miles per hour

80 miles in 4 hours = 20 miles per hour

and ask ourselves how we arrived at the speed, we find that we have, in each case, to divide the distance by the time. We can make a rule to fit all cases:

The average speed is the total distance divided by the total time.

But we must be careful to make sure that we use the right units throughout.

Example:

What is the speed in feet per second (ft/s) of a cyclist travelling 4 miles in 12 minutes?

As we want the speed in feet per second, we must change the miles to feet and the minutes to seconds.

$$4 \text{ miles} = 4 \times 5280 \text{ ft } (5280 \text{ ft} = 1 \text{ mile})$$

$$12 \text{ minutes} = 12 \times 60 \text{ seconds}$$

$$\text{Since Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\therefore \text{Speed} = \frac{4 \times 5280}{12 \times 60} \text{ ft/s}$$

$$= \frac{\overset{1}{\cancel{4}} \times \overset{88}{\cancel{5280}}}{\underset{3}{\cancel{12}} \times \cancel{60}} \text{ ft/s}$$

$$= \frac{88}{3} \text{ ft/s}$$

$$= 29\tfrac{1}{3} \text{ ft/s}$$

Notice that all the calculation is kept until the end. You will find quite often that you are able to cancel, and thus reduce the amount of calculating necessary.

EXERCISE 3E

Write the following rates in a simpler form:

1. 72 miles in 8 hours.

2. 5 lb of apples for 35p.

3. 4 ft of wood for 28p.

4. 3000 ft in 50 seconds.

5. 6,000 miles in half an hour.

6. Twelve teachers for 384 pupils.

7. Six wickets for 78 runs.

8. 3 lb of marmalade for $28\frac{1}{2}$p.

9. A journey by train from London to Bath (105 miles) took 180 minutes. What was the average speed of the train?

10. A through train to Italy leaves Calais at 3 p.m. and arrives in Milan at 9 a.m. the following morning. The distance between Calais and Milan is 636 miles. What is the average speed of the train, including stops, in mile/h?

11. What is a speed in feet per second of 60 mile/h? 30 mile/h? 45 mile/h?

12. Two boys were boasting about how fast they could cycle. "I have done 45 miles in $2\frac{1}{2}$ hours," said Taylor. "My best is 56 miles in $3\frac{1}{2}$ hours," said Baker. What were their average speeds in mile/h, and who was the faster on these performances?

13. An engineer works a 40-hour week and is paid £24·40. How much does he earn in one hour? If he works 3 hours overtime one week at double this rate, how much will he receive at the end of that week?

14. A surveyor, measuring for a new road, finds that the site rises 32 ft in the first $\frac{1}{4}$ mile, and 188 ft in the next $\frac{3}{4}$ mile. If he wishes to average the slope over the whole mile, what will it be, in inches rise, per yd of road?

15. If my dog runs 3 yd to my 1, and I take 90 minutes to do 5 miles, what is his average speed in mile/h?

16. A clock was 3 seconds fast at eight o'clock one morning and by six o'clock that evening was 27 seconds slow. When was it right, if it loses at a steady rate?

Time—Distance—Speed

Consider these easy examples:

Speed 40 mile/h, distance 120 miles, TIME = 3 hours (120 ÷ 40)
Speed 30 mile/h, distance 60 miles, TIME = 2 hours (60 ÷ 30)
Speed 30 mile/h, time 4 hours, DISTANCE = 120 miles (30 × 4)
Speed 70 mile/h, time 3 hours, DISTANCE = 210 miles (70 × 3)

It is not difficult to see that $\text{Time} = \dfrac{\text{Distance}}{\text{Speed}}$

$$\text{and Distance} = \text{Speed} \times \text{Time}$$

If we have any two of these three quantities, we can find the third.

Example:

An aircraft covered the distance from Gander to Shannon (2240 miles) at an average speed of 280 mile/h. How long did it take?

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

$$\therefore \text{Time} = \frac{2240}{280} \text{ hours} = \frac{\overset{8}{\overset{\cancel{32}}{\cancel{2240}}}}{\underset{1}{\underset{\cancel{4}}{\cancel{280}}}} \text{ hours}$$

$$= 8 \text{ hours}$$

Example:

A racing car completed a Grand Prix course in $2\frac{1}{2}$ hours at an average speed of 112 mile/h. How many miles did it cover?

$$\begin{aligned}\text{Distance} &= \text{Speed} \times \text{Time}\\ &= 112 \times 2\tfrac{1}{2} \text{ miles}\\ &= 280 \text{ miles}\end{aligned}$$

EXERCISE 3F

1. A passenger liner completed a voyage of 2140 nautical miles at an average speed of 20 knots. How long did she take in days and hours? (Knots = nautical miles per hour.)

2. A boy cycled 18 miles at a speed of 12 mile/h, rested for 30 minutes, then cycled a further 44 miles at a speed of 8 mile/h. If he started at 8 a.m., at what time did he arrive?

3. A train leaves London (Euston) for Manchester at 10 a.m. If it averages 55 mile/h and arrives at 1 p.m., how far is it from London to Manchester by rail?

4. If a tractor can draw a plough at an average speed of 3 mile/h (including turns) and a certain field is ploughed in $3\frac{1}{2}$ hours, how far has the tractor travelled in yards?

5. A dinghy sails at 6 mile/h in still water and travels 1 mile. How long does it take? If it then sails in the opposite direction at 2 mile/h for 1 mile, how long has it sailed in that direction? It has now travelled 2 miles. How long has it taken for both journeys? What is its average speed there and back?

6. A boy cycles 42 miles in 4 hours. How long will he take to cycle $73\frac{1}{2}$ miles at the same rate?

7. Mr. Smith averaged 20 mile/h in his car for $1\frac{1}{2}$ hours and then 36 mile/h for the next $2\frac{1}{2}$ hours. What was his average speed for the whole journey? (*Hint:* Find the distance for each part of the journey first.)

8. 12 German Deutsche marks are worth £1·32. How much is each mark worth?

9. Complete the missing spaces in this table:

Distance	*Time*	*Speed*
38 miles	3 hours	.... mile/h
.... ft	32 seconds	12 ft/s
185 miles	.... hours	25 mile/h
894 naut. miles	6 hours	.... knots
1100 miles	.... hours	225 mile/h
.... miles	2 minutes	88 ft/s

CHAPTER 4

DECIMALS

MOST OF US take it very much for granted that there are only 10 numerals. Look at this list of numbers:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19.

We see nothing strange in using the same numerals again after 9. But when we write "14" we use the "1" in this number to stand for something quite different to what is meant by the *first* 1 in the list. It means, of course, one TEN, just as in the number 145, the 1 stands for one HUNDRED, and the 4 for four TENS.

It is rather surprising, when you come to think of it. Why have we only 9 numerals and 0? Could we not have different signs to stand for 10, 11, 12, and so on? Well, of course, we could if we wished.

Let us invent two more, ↊ for 10, and ↋ for 11. What would our list look like now? The same as before up to 9: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and then ↊ for 10, ↋ for 11. But when we get to 12, we have to begin using our numerals in combination again. The old 12 will become the new 10, 13 becomes 11 and so on.

This is perhaps very unsettling. It is, you may think, bad enough to cope with the present numerals without new ones, but it does show that our system of counting is not the only one, or indeed the only useful one. There are nowadays thousands of machines—computers—which use only two numerals, 0 and 1, and yet do in minutes what it would take scientists and mathematicians months to work out.

Let us see what our numbers look like on such a machine:

Our numbers	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10
Computer numbers	0,	1,	10,	11,	100,	101,	110,	111,	1000,	1001,	1010 etc.

Since there are only two numerals in the binary scale (as it is called) the third number (our 2) has to be made by using 1 and 0 in combination, so we have:

Our number		*Binary*
1	=	1
2	=	10
4	=	100
8	=	1000

Can you say what 16 would be on the binary scale?

You will not be required to learn more just yet about other ways of counting. The point to remember is that our present system is a convention—and a very useful one—many hundreds of years old, and that we take for granted that when we write, say, 333, we do not mean $3 + 3 + 3$, or $3 \times 3 \times 3$ (as xyz *would* mean $x \times y \times z$ in algebra), but 3 hundreds, 3 tens and 3 units, and that, reading from the left each number is one-tenth of the preceding one.

Decimall Arithmetick.

computation is found by the consideration of such tenth or disme progression; that is, that it consisteth therein entirely, as shall hereafter appeare: Wee call this Treatise fitly by the name of Disme, whereby all accounts happening in the affaires of man, may be wrought and effected without fractions or broken numbers, as hereafter appeareth.

An extract from Stevin's book, La Disme: The Art of Tenths.

A mathematician called Stevin, writing in 1585, suggested that this convention ought to be used for fractions as well, and since that time what we call decimal fractions have been in general use. Decimals are fractions whose bottom part, or denominator, is a power of 10, that is, 10, 100, 1000, 10,000 and so on. You can see how this follows from our usual method of writing numbers if you look at this table:

Hundreds 100	*Tens* 10	*Units* 1	*Tenths* $\frac{1}{10}$	*Hundredths* $\frac{1}{100}$	*Thousandths* $\frac{1}{1000}$
3	3	3	3	3	3

Each number is ten times smaller than the one on its left. Notice that there is a double line where the whole numbers stop and the fractions begin. When we write such a number normally we use a DECIMAL POINT, and write the number 333·333. In other countries other methods are used. (The French use a comma, the Americans put the dot *on* the line.) The decimal point shows where the whole numbers finish and the fractions begin.

1·4 means $1\frac{4}{10}$ or $1\frac{2}{5}$.
3·06 means $3\frac{6}{100}$.
4·7 means $4\frac{7}{10}$.
5·1 means $5\frac{1}{10}$.
2·35 (say, "Two point three, five") means $2 + \frac{3}{10} + \frac{5}{100}$, or $2\frac{35}{100}$.
9·003 (say, "Nine point nought, nought, three") means $9\frac{3}{1000}$.
3·126 means $3 + \frac{1}{10} + \frac{2}{100} + \frac{6}{1000}$ or $3\frac{126}{1000}$.
0·07 means $\frac{7}{100}$.

Let us draw up a table, showing what some other numbers mean:

	Hundreds 100	*Tens* 10	*Units* 1	*Tenths* $\frac{1}{10}$	*Hundredths* $\frac{1}{100}$	*Thousandths* $\frac{1}{1000}$
1·23 =			1	2	3	
40·06 =		4	0	0	6	
1·011 =			1	0	1	1
345·91 =	3	4	5	9	1	

EXERCISE 4A

1. Make a table like the one above, and write in it the following numbers: $\frac{3}{10}$, $\frac{7}{100}$, $\frac{86}{100}$, $2\frac{1}{10}$, 5·94, 78·12, 935·307, nine thousandths, twelve hundredths (What are ten hundredths?).

2. Write as decimals, $1\frac{3}{10}$, $5\frac{7}{10}$, $134\frac{1}{10}$, $\frac{3}{5}$ (How many *tenths* is this?), $3\frac{1}{5}$, $12\frac{23}{100}$, $9\frac{2}{1000}$, $2\frac{7}{100}$, $67\frac{34}{1000}$, $1\frac{203}{1000}$.

3. Write as mixed numbers: 5·1, 3·3, 7·9, 0·5, 4·8, 4·18, 18·33, 1·625.

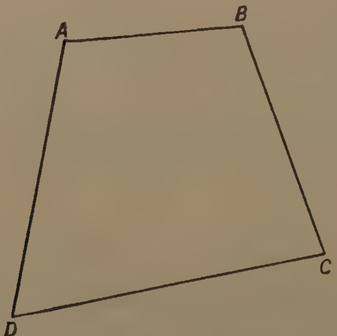
4. With a ruler marked in tenths of an inch, draw a line, and from one end mark off the lengths 1·6, 2·3, 2·9, 3·4, and 0·7 inches.

5. Write as decimals $\frac{27}{1000}$, $\frac{424}{1000}$, $\frac{32}{10}$, $\frac{621}{100}$, $\frac{1007}{1000}$, thirty-six hundredths, $\frac{14278}{1000}$.

6. Draw a line XY, 3·9 in long, and mark any point A on the line. Measure XA and AY. Add the two lengths together.

7. Write as fractions in their lowest terms: 0·02, 0·4, 0·25, 0·75, 0·125, 0·72, 0·875.

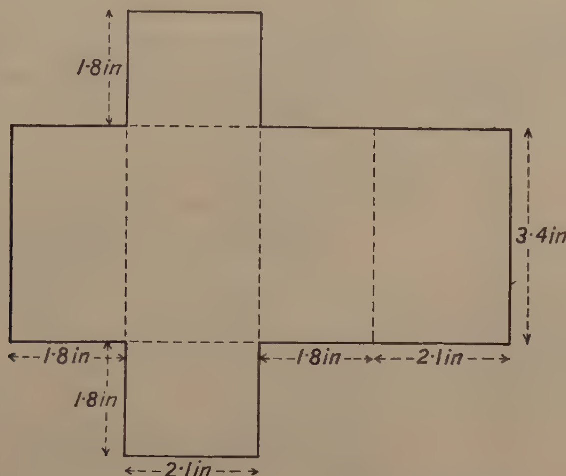
8.



Measure AB, BC, CD and DA in inches and decimals of an inch. By adding these lengths find the perimeter of the figure.

9. Write the following fractions with denominator 10, and then as decimals: $\frac{1}{2}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{1}{5}$.

10. Write the following fractions with denominator 100, and then as decimals: $\frac{1}{25}$, $\frac{3}{25}$, $\frac{17}{25}$, $\frac{1}{20}$, $\frac{13}{20}$.



11. Draw this shape carefully to the dimensions given, on a piece of thin card. Cut it out and fold it at the dotted lines to form a box. The edges can be held together with strips of transparent self-adhesive tape.

12. If 1 French franc = 100 centimes, write the following amounts in francs and centimes: 1·72 francs; 34·07 francs; 12·96 francs.

13. A boy cycles 5·7 miles west from his house (*H*) to a place *A*. He then cycles 7·4 miles east from *A* to place *B*. Finally he cycles 3·4 miles west to *C*. Using a scale of 1 in to 1 mile draw a plan of his afternoon's cycling and measure how far he is from *H* at the end of his journey.

In No. 14-18 write the sum of the numbers as decimals.

14. $30 + 6 + \frac{3}{10} + \frac{4}{100}$.

15. $20 + 8 + \frac{8}{10} + \frac{2}{100}$.

16. $100 + 40 + 9 + \frac{7}{10} + \frac{2}{100} + \frac{8}{1000}$.

17. $80 + 4 + \frac{6}{100}$.

18. $5 + \frac{4}{10} + \frac{7}{1000}$.

19. Draw a triangle *ABC*, using your ruler and compasses, with *AB* = 3·8 in, *BC* = 2·4 in, *CD* = 1·9 in.

20. Which is bigger, 9 or 0·9; 10 or 0·11; 27 or 0·8999?

Learn the following fractions and their decimal equivalents by heart:

$\frac{1}{2} = 0{\cdot}5$ $\qquad$ $\frac{1}{8} = 0{\cdot}125$ $\qquad$ $\frac{5}{8} = 0{\cdot}625$

$\frac{1}{4} = 0{\cdot}25$ $\qquad$ $\frac{3}{8} = 0{\cdot}375$ $\qquad$ $\frac{7}{8} = 0{\cdot}875$

$\frac{3}{4} = 0{\cdot}75$

Addition and Subtraction

Decimal fractions can be added or subtracted in the same way as whole numbers. *It is important to keep the decimal points underneath one another.*

Example: Add together 17·28, 6·94, 0·06.

$$\begin{array}{r} 17{\cdot}28 \\ 6{\cdot}94 \\ 0{\cdot}06 \\ \hline 24{\cdot}28 \\ \hline \end{array}$$

(Check by adding both up and down.)

Notice that we carry across the decimal point in the same way as we carry from units to tens.

Example: Find the value of 27·34 − 17·85.

$$\begin{array}{lr} \text{(i)} & 27{\cdot}34 \\ \text{(ii)} & 17{\cdot}85 \\ \cline{2-2} \text{(iii)} & 9{\cdot}49 \\ \cline{2-2} \end{array}$$

(Check by adding (ii) and (iii) together, which should give (i).)

Example: Find the value of 42·16 − 12·857.

$$\begin{array}{r} 42{\cdot}16 \\ 12{\cdot}857 \\ \hline 29{\cdot}303 \\ \hline \end{array}$$

(42·16 can, if required, be written as 42·160. There are no thousandths in the top number, so the 6 hundredths must be split into 5 hundredths and 10 thousandths. 7 from 10 leaves 3, 5 from 5 leaves 0 and so on.)

EXERCISE 4B

Write down and add the following:

1. $\begin{array}{r} 27{\cdot}23 \\ 13{\cdot}45 \\ \hline \end{array}$

2. $\begin{array}{r} 16{\cdot}71 \\ 12{\cdot}25 \\ \hline \end{array}$

3. $\begin{array}{r} 5{\cdot}62 \\ 17{\cdot}33 \\ \hline \end{array}$

4. $\begin{array}{r} 12{\cdot}92 \\ 14{\cdot}05 \\ \hline \end{array}$

5. $\begin{array}{r} 17{\cdot}1 \\ 34{\cdot}75 \\ \hline \end{array}$

6. $\begin{array}{r} 129{\cdot}04 \\ 16{\cdot}8 \\ \hline \end{array}$

7. $\begin{array}{r} 94{\cdot}632 \\ 4{\cdot}81 \\ \hline \end{array}$

8. $\begin{array}{r} 56{\cdot}71 \\ 0{\cdot}089 \\ \hline \end{array}$

9. $\begin{array}{r} 18{\cdot}31 \\ 0{\cdot}065 \\ 132{\cdot}495 \\ \hline \end{array}$

10. $\begin{array}{r} 3{\cdot}2 \\ {\cdot}999 \\ 17{\cdot}001 \\ \hline \end{array}$

11. $\begin{array}{r} 45{\cdot}36 \\ 107{\cdot} \\ 0{\cdot}8 \\ \hline \end{array}$

12. $\begin{array}{r} 14{\cdot}4 \\ 3{\cdot}35 \\ 137{\cdot}072 \\ \hline \end{array}$

Write down the following sums and subtract the second number from the first:

13. 37·6 25·2	**15.** 12·3 8·8	**17.** 54·6 13·08	**19.** 310· 305·62
14. 92·8 34·3	**16.** 19·09 12·32	**18.** 73·5 27·63	**20.** 407·03 215·679

Combined Calculations

Example:

Find the value of 14·27 — 3·46 + 12·701 — 5·85.

This sum has a number of instructions: "Start with 14·27, take from it 3·46 and 5·85, and add to it 12·701."

Start by adding 12·701 to 14·27:

$$\begin{array}{r} 14{\cdot}27 \\ \underline{12{\cdot}701} \\ 26{\cdot}971 \end{array}$$

Next *add* together 3·46 and 5·85 (in order that they can be subtracted together):

$$\begin{array}{r} 3{\cdot}46 \\ \underline{5{\cdot}85} \\ 9{\cdot}31 \end{array}$$

Then do the subtraction:

$$\begin{array}{r} 26{\cdot}971 \\ \underline{9{\cdot}31} \\ \underline{17{\cdot}661} \end{array}$$

EXERCISE 4C

Find the value of:

1. 3·4 + 5·6 — 3·7 — 5·1.

2. 18·45 + 6·2 — 8·03 — 12·1.

3. 9·601 — 1·43 + 17·21 — 12·084.

4. 1·65 + 18·91 — 7·32 — 0·405.

5. 6·1 + 45 — 13·697 — 18·04.

6. A thermometer reads 11·4° C at 9 a.m. If the temperature rises 8·7° C in the next five hours, what will the thermometer reading be at 2 p.m.?

7. A jug 8·7 in deep contains milk to a depth of 5·65 in. How far is the surface of the milk from the top of the jug?

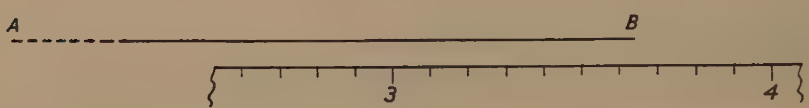
8. A dictionary is 2·9 in thick and each cover is 0·15 in thick. How thick is the dictionary without covers?

9. The sides of a triangle are 5·35 in, 3·8 in and 4·97 in. What is the perimeter of the triangle?

10. An earthenware drainpipe has an external diameter of 6·7 in. If the earthenware is 0·43 in thick, what is the internal diameter?

Accuracy of Measurement

When we measure lengths it is impossible to be absolutely accurate. If we measure a line with a ruler, using the $\frac{1}{10}$ of an inch measure we can be accurate to one-tenth of an inch. With practice it is even possible to be quite accurate to $\frac{1}{100}$ of an inch.



Magnified section of ruler determining length of line as 3·64 *in approx.*

This shows a magnified part of a ruler, divided into tenths of an inch measuring the length of a line. If the beginning of the line (A) is placed against 0 on the ruler, you can see that B, the end of the line, measures 3 whole inches $+ \frac{6}{10}$ in $+$ a little more. If we mentally divide the space between 3·6 and 3·7 into 10 parts, each part will be $\frac{1}{10}$ of $\frac{1}{10}$ of an inch, i.e. $\frac{1}{100}$ in. In this case we can estimate that the line ends after 4 of these, or $\frac{4}{100}$. So our best guess as to the length of the line will be $3 + \frac{6}{10} + \frac{4}{100}$, or 3·64 in.

But that is as far as we can go with a ruler. If we were measuring a metal rod of the same length as this line, we could measure fairly accurately to $\frac{1}{1000}$ of an inch with a micrometer and we might find that it showed that the length was 3·643 in. And there are instruments which make even a micrometer look very inaccurate and clumsy, instruments using microscopes and rays of light. On the other hand, while it would be important to measure a part of a motor-car engine to the nearest thousandth of an inch, to ensure a perfect fit, it would not be at all necessary in the case, say, of a long racecourse.

Different cases need differing degrees of accuracy and when we write measurements of length or area, we show how accurate our measurement is.

Suppose we find as the result of a calculation that a metal rod which we will call XY, should be 4·237 in long. If we wish to draw a line to represent XY with a ruler we could certainly get the tenths right. The line XY would then be 4·2 in long, *correct to* 1 *place of decimals.*

If with more accurate instruments we could draw it correct to the nearest $\frac{1}{100}$ of an inch, XY would then be 4·24 in long (since ·24 is nearer to ·237 —only $\frac{3}{1000}$ of an inch away in fact—than is ·23, which is $\frac{7}{1000}$ of an inch away), and the length (4·24) would then be *correct to 2 places of decimals.*

So 4·237 becomes 4·24 correct to 2 decimal places
and 4·2 correct to 1 decimal place.

The word decimal is often left out and we say "correct to ...... places."

The number 5·47286
will become 5·4729 correct to 4 places,
5·473 correct to 3 places,
5·47 correct to 2 places,
5·5 correct to 1 place.

What would 6·35 be correct to 1 place? It is exactly halfway between 6·3 and 6·4 (as 35 is halfway between 30 and 40). We can, if we are working to one place, put the second figure in brackets—6·3(5), or we can go up or down to the even number—a method used in many laboratories.

Thus 6·35 becomes 6·4 correct to 1 place.
6·45 becomes 6·4 correct to 1 place.
6·55 becomes 6·6 correct to 1 place.

But of course 6·451 correct to one place would become 6·5 (·451 being nearer to ·500 than to ·400).

EXERCISE 4D

Write the following numbers, correct to (a) 1 place of decimals; (b) 2 places of decimals; (c) the nearest whole number.

1. 3·168 **4.** 1·049 **7.** 15·25

2. 5·042 **5.** $\frac{13}{20}$ **8.** 9·999

3. 8·887 **6.** 38·52548

Write the following numbers correct to 4 places, 3 places, 2 places and to 1 place of decimals:

9. 3·43182 **11.** 1·007386

10. 2·08316 **12.** 2·97391

13. Add together 7·342 + 1·00876 + 0·0793 + 14·70085 and give the result correct to (a) 4 places, (b) 3 places, (c) 2 places.

14. Work out 1·0057 − 12·4208 + 19·878 − 3·04 and give the result correct to 3 places.

Multiplication

One of the great advantages of the decimal system is that multiplication or division by 10, 100, 1,000 or any power of 10 is extremely easy. Consider the following example:

$$\frac{7}{10} \times 10 = \frac{70}{10}$$
$$= 7$$

So to multiply 0·7 by 10 it is only necessary to move the decimal point one place to the right.

$$2{\cdot}84 \times 10 = 28{\cdot}4$$
$$1{\cdot}507 \times 10 = 15{\cdot}07$$

Since multiplying by 100 is the same as multiplying by 10 × 10,

$$3{\cdot}45 \times 100 = 34{\cdot}5 \times 10$$
$$= 345$$

To multiply a decimal fraction by 10, *move the decimal point one place to the right.*

To multiply a decimal fraction by 100, *move the decimal point two places to the right.*

To multiply a decimal fraction by 1,000 *move the decimal point three places to the right;*

and so on.

Similarly:

$$8{\cdot}6 \div 10 = \frac{86}{10} \div 10$$
$$= \frac{86}{10} \times \frac{1}{10}$$
$$= \frac{86}{100}$$
$$= 0{\cdot}86$$

and the rule for dividing by 10 is to move the decimal point one place to the left, for dividing by 100 move it two places to the left, and so on.

Note: Although it is useful to think of moving the decimal point, it is as well to remember that it is really the figures that are being moved in place value—up when multiplying, down when dividing.

EXERCISE 4E

Multiply each of the following numbers by 10, by 100 and by 1 000:

1. 3·2

2. 4·38

3. 0·76

4. 0·025

5. 15·003

Divide each of the following numbers by 10, by 100 and by 1 000:

6. 70

7. 425·6

8. 1·234

9. 0·05

10. 58·37

Complete the following:

11. 3·1 × ...... = 31

12. 0·63 × ...... = 63

13. 14·06 ÷ ...... = 1·406

14. 19·001 × ...... = 19,001

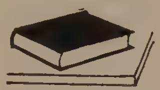
15. 0·0084 × ...... = 8·4

16. 0·0973 ÷ ...... = 0·00973

17. From a ball of string 9 ft long, 10 pieces, each 10·7 in long, are cut off. How long is the remaining piece?

18. How many tenths are there in 0·25?

19. A 100-leaf book with covers is 1·06 in thick. If the covers are each 0·12 in thick, how thick is each leaf?



20. Write down, without written working if you can, the answers to the following sums:

(a) 0·3 × 20;

(b) 0·03 × 40;

(c) 0·05 × 600.

Multiplication by Integers

(An integer is a whole number.)

Example: Multiply 0·7 by 11.

This is the same as saying $\frac{7}{10} \times 11$

$= \frac{77}{10}$

$= 7{\cdot}7$

Example: Multiply 1·62 by 8.

$$\begin{array}{r} 1{\cdot}62 \\ 8 \\ \hline 12{\cdot}96 \\ \hline \end{array}$$

Put the 8 under the right-hand digit of 1·62. The decimal point of the answer comes under the decimal point in 1·62.

Example: Multiply 0·039 76 by 12.

$$\begin{array}{r} 0{\cdot}03976 \\ 12 \\ \hline 0{\cdot}47712 \\ \hline \end{array}$$

Notice that the last digit in the multiplier comes under the last digit in 0·03976, and once again the decimal points are underneath each other.

Example: Multiply 43·62 by 120.

This is the same as 43·62 × 12 × 10

Multiply by 12,

$$\begin{array}{r} 43{\cdot}62 \\ 12 \\ \hline 523{\cdot}44 \end{array}$$

and then by 10 5234·4

EXERCISE 4F

Find the values of:

1. $3{\cdot}2 \times 4$
2. $5{\cdot}7 \times 6$
3. $15{\cdot}08 \times 5$
4. $12{\cdot}34 \times 9$
5. $0{\cdot}019 \times 3$
6. $0{\cdot}904 \times 7$
7. $13{\cdot}51 \times 12$
8. $4{\cdot}008 \times 11$
9. $7{\cdot}84 \times 8$
10. $4{\cdot}1006 \times 6$
11. $0{\cdot}004 \times 6$
12. $0{\cdot}09 \times 12$
13. $1{\cdot}2 \times 12$
14. $0{\cdot}4 \times 30$
15. $0{\cdot}6 \times 50$
16. $0{\cdot}06 \times 5{,}000$
17. $0{\cdot}34 \times 100$
18. $0{\cdot}7 \times 110$
19. $1{\cdot}09 \times 700$
20. $3{\cdot}28 \times 90$

Short Division of Decimals

To divide a decimal by an integer of 12 or under, set the sum out as an ordinary short division sum:

Example: $38{\cdot}79 \div 9$

$$\begin{array}{r} 9\overline{)38{\cdot}79} \\ \hline 4{\cdot}31 \end{array}$$

Keep the decimal points underneath each other.

Example: $0{\cdot}008349 \div 11$

$$\begin{array}{r} 11\overline{)0{\cdot}008349} \\ \hline 0{\cdot}000759 \end{array}$$

Example: $2{\cdot}5 \div 4$

$$\begin{array}{r} 4\overline{)2{\cdot}500} \\ \hline 0{\cdot}625 \end{array}$$

EXERCISE 4G

Work out:

1. $5{\cdot}4 \div 2$
2. $8{\cdot}7 \div 3$
3. $1{\cdot}08 \div 12$
4. $37{\cdot}73 \div 11$
5. $0{\cdot}504 \div 6$
6. $1{\cdot}75 \div 5$
7. $14 \div 4$
8. $0{\cdot}091 \div 7$
9. $6{\cdot}8 \div 8$
10. $0{\cdot}7137 \div 9$
11. $160{\cdot}008 \div 12$
12. $0{\cdot}4 \div 8$
13. $3 \div 6$
14. $15 \div 4$
15. $0{\cdot}75 \div 12$

Multiplication By Any Number

Suppose we wish to multiply 0·06 by 0·7. As fractions, this sum would read $\frac{6}{100} \times \frac{7}{10} = \frac{42}{1000}$ or 0·042. We multiply the 6 by the 7, and there are as many decimal places in the answer as the total number in both the original numbers (in this case three). The rule is:

Multiply the numbers together, ignoring the decimal points: add together the number of decimal places in the two numbers, count off this number from the right and put in the decimal point.

Example: 14·62 × 3·4

Multiply the numbers ignoring the decimal points:

$$\begin{array}{r} 1462 \\ \underline{34} \\ 5848 \\ \underline{43860} \\ 49708 \end{array}$$

There are 3 decimal places (two in 14·62, one in 3·4).

Count these off from the right: $49{\cdot}\overset{(3)}{7}\,\overset{(2)}{0}\,\overset{(1)}{8}$.

Example: 10·104 × 0·0032

$$\begin{array}{r} 10104 \\ \underline{32} \\ 20208 \\ \underline{303120} \\ 323328 \end{array}$$

There are (3 + 4) or 7 decimal places; counting 7 from the right we find we have to insert an extra zero before inserting the decimal point:

0·0323328

Example: 0·025 × 0·04

$$\begin{array}{r} 25 \\ \underline{4} \\ 100 \end{array}$$

∴ the answer is 0·00100

Note: Do not cross off the noughts in the product before counting back from the right.

It is sometimes useful to make a rough check *before* doing the sum to be sure that the decimal point is in the right place. 207·3 × 0·87, for example, is approximately 200 × ·9, or 180.

EXERCISE 4H

Find the value of:

1. 0·4 × 0·2
2. 0·6 × 0·4
3. 4 × 0·8
4. 0·7 × 3·2
5. 5·4 × 0·3
6. 0·5 × 0·04
7. 1·34 × 2·5
8. 19·1 × 0·43
9. 1·06 × 7·2
10. 34·51 × 12·97
11. 7·64 × 14·38
12. 2·06 × 8·08
13. 370 × 0·04
14. 0·39 × 1200
15. 7·24 × 1·03
16. 6,200 × 0·317
17. 1·2 × 23·1052
18. 11·78 × 1·67
19. 1·25 × 0·001 03
20. 70·94 × 1·0602

Division By Any Number

There are a number of methods of dividing one decimal by another. One of the simplest is to make the divisor (the number we are dividing by) into a whole number by multiplying it by a power of 10, i.e. 10, 100, 1,000. Supposing we have to divide 76·59 by 3·7. If we multiply 3·7 by 10 it will become 37, a whole number. But, if we are to get an answer of the right size, we must multiply 76·59 by 10 also. Our sum now reads 765·9 ÷ 37.

Set the sum out in the form of a long division sum and put the decimal point of the quotient immediately above that of the dividend:

$$
\begin{array}{r}
\cdot \\
37\overline{)765{\cdot}9}
\end{array}
$$

Now work out the sum in the usual way:

Rough check: $\frac{800}{40} = 20$

$$
\begin{array}{r}
20{\cdot}7 \\
37\overline{)765{\cdot}9} \\
\underline{74\phantom{5{\cdot}9}} \\
25{\cdot}9 \\
\underline{25{\cdot}9} \\
\cdots
\end{array}
$$

Answer: 20·7

Example: 2·3085 ÷ 0·045

0·045 × 1000 = ...45

2·3085 × 1000 = 2308·5

Rough check: $\frac{2500}{50} = 50$

$$
\begin{array}{r}
51{\cdot}3 \\
45\overline{)2308{\cdot}5} \\
\underline{225\phantom{8{\cdot}5}} \\
58\phantom{{\cdot}5} \\
\underline{45\phantom{{\cdot}5}} \\
13\phantom{{\cdot}5}
\end{array}
$$

Answer: 51·3

EXERCISE 4J

Find the value of:

1. 36 ÷ 0·9
2. 7·2 ÷ 0·008
3. 720 ÷ 0·8
4. 2·88 ÷ 1·2
5. 3·454 ÷ 0·11
6. 127·16 ÷ 0·022
7. 56·472 ÷ 0·104
8. 1·65242 ÷ 20·3
9. 0·567 ÷ 0·126
10. 244·92 ÷ 31·4
11. 1782 ÷ 0·033
12. 26·104 ÷ 1·004
13. 112·596 ÷ 13·2
14. 0·0073 ÷ 7·3
15. 6·208 ÷ 19·4
16. 144·29 ÷ 307
17. 46·11 ÷ 0·87
18. 4·173 ÷ 0·039
19. 57·456 ÷ 18·9
20. 271·2 ÷ 9·04

Decimals and Vulgar Fractions

When we write a fraction like, for example, $\frac{2}{5}$, we mean that this is the result obtained by dividing 2 by 5. But if we do this as a decimal sum $5\overline{)2{\cdot}0}$, we obtain the result 0·4. So $0{\cdot}4 = \frac{2}{5}$. Any vulgar fraction can be changed to a decimal fraction by dividing the numerator (the top part of the fraction) by the denominator (the bottom part). Thus $\frac{3}{4} = 3 \div 4 = 0{\cdot}75$, and $\frac{3}{8} = 3 \div 8 = 0{\cdot}375$.

However if we try to change $\frac{2}{3}$ into a decimal fraction: $\begin{array}{r} 3\overline{)2{\cdot}000} \\ \overline{0{\cdot}666} \end{array}$ we never get to the end of the sum, however long the calculation goes on.

Such a fraction is called a RECURRING decimal, and is written $0{\cdot}\dot{6}$ ("Nought point six recurring"). Sometimes a group of numbers recurs in the quotient, as in the following example:

$$\frac{2}{7} = 2 \div 7$$

$$\begin{array}{l} 7\overline{)2{\cdot}000000000000} \\ \quad {\cdot}285714285714 \text{ etc.} \end{array}$$

The group 285714 recurs. The quotient is then written

$$0{\cdot}\dot{2}85\ 71\dot{4}$$

with a dot over the first and last recurring number. In practice it is usually better to use a vulgar fraction than a recurring decimal. If decimals are used, give the number to as many decimal places as you think justified by the sort of measurements you are using. With a ruler, you could perhaps give the above fraction as 0·29 correct to 2 places, by careful measurement.

EXERCISE 4K

Convert the following fractions to decimal fractions. If the answer is a recurring decimal, give the answer to 3 places (which means that they must be worked to 4 places of decimals, then corrected).

1. $\frac{1}{3}$	**3.** $\frac{1}{9}$	**5.** $\frac{5}{9}$	**7.** $\frac{3}{7}$	**9.** $\frac{5}{11}$
2. $\frac{5}{6}$	**4.** $\frac{2}{9}$	**6.** $\frac{1}{7}$	**8.** $\frac{4}{7}$	**10.** $\frac{7}{13}$

CHAPTER 5

FRACTIONS II

IT IS EASY to see that $\frac{3}{4}$ is bigger than $\frac{1}{8}$, but it is not so simple to tell at first sight whether $\frac{7}{10}$ is greater or less than $\frac{11}{15}$.

$$\frac{7}{10} = \frac{21}{30}$$
$$\text{and } \frac{11}{15} = \frac{22}{30}$$

and so we see that $\frac{11}{15}$ is just the greater, by $\frac{1}{30}$.

What we have done here is to arrange that both fractions have the same number at the bottom. We say that we have expressed both fractions with a COMMON DENOMINATOR.

To choose a common denominator, we must find a number into which both the given denominators (here 10 and 15) will divide exactly. There will, of course, be many such numbers; in this case, 30, 60, 90 etc. all contain both 10 and 15 an exact number of times, and we could use any of these, but 30 is the smallest such number, and therefore the easiest to work with. It is called the LOWEST COMMON DENOMINATOR, or lowest common MULTIPLE (L.C.M. for short).

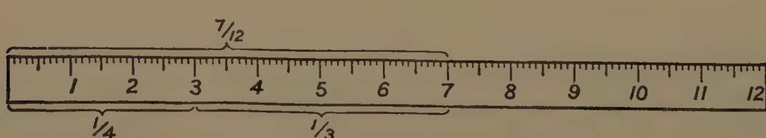
EXERCISE 5A

Find the L.C.M. of:

1. 3 and 6.	**4.** 4 and 12.	**7.** 8 and 12.	**10.** 6, 8 and 10.
2. 4 and 6.	**5.** 5 and 12.	**8.** 4 and 10.	
3. 6 and 9.	**6.** 10 and 12.	**9.** 15 and 20.	

Addition

When we want to add $\frac{1}{4}$ ft and $\frac{1}{3}$ ft we can say



$$\tfrac{1}{4}\text{ ft} = 3\text{ in} = \tfrac{3}{12}\text{ ft}$$
$$\tfrac{1}{3}\text{ ft} = 4\text{ in} = \tfrac{4}{12}\text{ ft}$$

so $\frac{1}{4}$ ft $+ \frac{1}{3}$ ft $= 7$ in $= \frac{7}{12}$ ft.

Put another way, we can say

$$\tfrac{1}{4} + \tfrac{1}{3} = \tfrac{3}{12} + \tfrac{4}{12} = \tfrac{7}{12}$$

Thus, to add up fractions with different denominators we first express each of them with a common denominator. The same applies to subtraction:

$$\tfrac{7}{8} - \tfrac{1}{6} = \tfrac{21}{24} - \tfrac{4}{24} = \tfrac{17}{24}$$

EXERCISE 5B

Evaluate:

1. $\frac{1}{2} + \frac{1}{4}$
2. $\frac{1}{4} + \frac{2}{3}$
3. $\frac{2}{3} - \frac{1}{4}$
4. $\frac{5}{9} + \frac{5}{6}$
5. $\frac{7}{10} - \frac{1}{2}$
6. $1 - \frac{7}{13}$ (remember that $1 = \frac{13}{13}$)
7. $1 - \frac{19}{24}$
8. $\frac{3}{8} + \frac{8}{15}$
9. $\frac{3}{10} + \frac{11}{25}$
10. $\frac{5}{8} + \frac{7}{12} - \frac{11}{18}$

Mixed Numbers

When dealing with mixed numbers, it is best to treat the whole numbers and the fractions separately. For example, to add $4\frac{3}{8}$ to $7\frac{11}{12}$, we first say 4 and 7 make 11, and then

$$\tfrac{3}{8} + \tfrac{11}{12} = \tfrac{9}{24} + \tfrac{22}{24} = \tfrac{31}{24} = 1\tfrac{7}{24}$$

hence the total is $11 + 1\frac{7}{24} = 12\frac{7}{24}$

We write the work like this:

$$4\tfrac{3}{8} + 7\tfrac{11}{12} = 11\frac{9+22}{24}$$

$$= 11\tfrac{31}{24} = 12\tfrac{7}{24}$$

Similarly, $$4\tfrac{3}{8} - 2\tfrac{1}{12} = 2\frac{9-2}{24} = 2\tfrac{7}{24}$$

If, however, we try to do this with the sum

$$4\tfrac{3}{8} - 2\tfrac{7}{12} = 2\frac{9-14}{24}$$

we run into a difficulty; we cannot take 14 from 9. To get round this problem, we must use the simple fact that $\frac{24}{24} = 1$

The number 2 which we have in front of the fraction can be written as

$1 + \frac{24}{24}$ (because this is the same as $1 + 1$)

and we now have $$2\frac{9-14}{24} = 1 + \frac{24}{24} + \frac{9-14}{24}$$

$$= 1 + \frac{24+9-14}{24}$$

$$= 1 + \frac{33-14}{24} = 1\tfrac{19}{24}$$

Here is another example of this kind of problem, showing you how to set out the work:

$$4\tfrac{2}{5} - 1\tfrac{5}{8} = 3\frac{16-25}{40}$$
$$= 2\frac{40+16-25}{40}$$
$$= 2\tfrac{31}{40}$$

EXERCISE 5C

Evaluate:

1. $2\frac{1}{2} + 3\frac{1}{6}$

2. $5\frac{1}{4} + 8\frac{3}{8}$

3. $7\frac{3}{4} + 4\frac{5}{6}$

4. $2\frac{7}{12} + 3\frac{5}{8}$

5. $6\frac{3}{8} - 4\frac{1}{4}$

6. $9\frac{3}{5} - 6\frac{3}{10}$

7. $18\frac{1}{3} - 7\frac{2}{3}$

8. $5\frac{1}{6} - 3\frac{7}{9}$

9. $3\frac{1}{6} + 7\frac{3}{8} + 4\frac{5}{12}$

10. $8\frac{3}{8} + 5\frac{5}{6} - 7\frac{2}{5}$

How to Find the L.C.M.

For most fractional work the L.C.M. is easy enough to guess, but occasionally we have to add or subtract fractions with very large denominators, when it is impossible to guess the L.C.M. Thus we must have a method of finding the L.C.M. when we need it.

To do this, we use the process of expressing numbers in prime factors which we met in Chapter 1. For example, to find the L.C.M. of 360 and 528, we say

$$360 = 2^3 \times 3^2 \times 5$$
$$528 = 2^4 \times 3 \times 11$$

Now any number into which both 360 and 528 divide exactly must contain the factors 5 and 11, the factor 3 at least twice, and the factor 2 at least four times: thus the *lowest* common multiple is

$$2^4 \times 3^2 \times 5 \times 11$$
$$= 7920$$

To find how many times 360 divides into 7920 we can of course do a long division sum, but it is easier to compare the prime factors of 360 and 7920.

$$\tfrac{7920}{360} = \frac{2^4 \times 3^2 \times 5 \times 11}{2^3 \times 3^2 \times 5}$$

In this fraction, 3^2 and 5 will cancel, and so will 2^3, leaving only 2×11 at the top of the fraction, thus:

$$\tfrac{7920}{360} = 2 \times 11 = 22$$

Similarly, $$\tfrac{7920}{528} = \frac{2^4 \times 3^2 \times 5 \times 11}{2^4 \times 3 \times 11} = 3 \times 5 = 15$$

Using this, we can evaluate

$$\frac{17}{360} + \frac{31}{528}$$
$$= \frac{17 \times 22}{7920} + \frac{31 \times 15}{7920}$$
$$= \frac{374 + 465}{7920}$$
$$= \frac{839}{7920}$$

EXERCISE 5D

1. Find the L.C.M. of 180 and 168. How many times does each number divide into this L.C.M.? Use your result to evaluate $\frac{11}{180} + \frac{65}{168}$.

2. Repeat Question **1** for the numbers 63 and 273. Hence evaluate $\frac{22}{63} - \frac{10}{273}$.

3. Find the L.C.M. of 4725 and 12375.

4. Evaluate $\frac{91}{600} + \frac{119}{180}$.

Square Root of Numbers in Index Form

We have already seen that 5^2 means $5 \times 5 = 25$: similarly $(3^3)^2$ means

$$3^3 \times 3^3 = 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6$$

that is, $$27^2 = 729$$

This shows that to square a number which is written in *index form*, we *double the index*. Since $5^2 = 25$, we say that 5 is the *square root* of 25. Similarly, 6 is the square root of 36, and 10 is the square root of 100. Also, the square root of 729 is 27; that is, the square root of 3^6 is 3^3.

To find the square root of a number written in index form, we *halve* the index. Thus by the prime factor process, we find that

$$4096 = 2^{12}$$

so the square root of 4096 is 2^6, or 64.

We use the symbol $\surd$ to denote a square root, and we write

$$\surd 4096 = 64$$

The number 7056 can be written in prime factors as

$$2^4 \times 3^2 \times 7^2$$

and so its square root is $2^2 \times 3 \times 7 = 84$

Thus $\surd 7056 = 84$

EXERCISE 5E

Evaluate:

1. $\surd 81$
2. $\surd 121$
3. $\surd 4$
4. $\surd 1$
5. $\surd 784$
6. $\surd 2916$
7. $\surd 6561$
8. $\surd 3025$
9. $\surd 2401$
10. $\surd 1764$

Brackets

You may have noticed that we have occasionally used brackets ( ) in order to make quite clear what a certain collection of numbers means. Brackets serve as a kind of basket. On the preceding page we wrote $(3^3)^2$ to show that for the moment we wanted to think of 3^3 as a single number (27, in fact) which was to be squared.

Brackets are very useful in this way, and should always be used if we think there could be any doubt as to what we mean. Consider for example the expression

$$\sqrt{49} \times 36$$

Does this mean we are to multiply 49 by 36 and then find the square root? Or does it mean that we are to find the square root of 49 and then multiply the result by 36? Brackets help to make it clear: if we mean the first thing, we would write $\sqrt{(49 \times 36)}$, and if we mean the second we would write $(\sqrt{49}) \times 36$.

You should check for yourself that the first expression comes to 42 and the second to 252. In fact, when we see a bracket, we must take it to mean that the *sum inside the bracket must be worked out before anything else.*

Example:

$$2\tfrac{1}{2} \times (\tfrac{1}{3} + \tfrac{1}{5}) = 2\tfrac{1}{2} \times \left(\frac{5+3}{15}\right)$$
$$= \tfrac{5}{2} \times \tfrac{8}{15}$$
$$= \tfrac{4}{3}$$
$$= 1\tfrac{1}{3}$$

Sometimes it is rather difficult to express exactly what you mean even by using brackets. To deal with such a case, mathematicians have agreed that, when there is any doubt, multiplication and division are done before addition and subtraction. (When there is no real reason for a rule, except that people have agreed to do it always in that way, the rule is called a *convention.*) Thus:

$$\tfrac{1}{3} + \tfrac{3}{8} \times \tfrac{2}{3}$$

implies that $\frac{3}{8}$ must be multiplied by $\frac{2}{3}$ *first*, giving the result $\frac{1}{4}$, and *then* $\frac{1}{3}$ must be added to $\frac{1}{4}$, giving the answer $\frac{7}{12}$. But it could have been made quite clear by the use of brackets to indicate that the multiplication should be done first. It would not be so easy, however, in this case:

$$\tfrac{2}{3} - \tfrac{1}{4} \times (\tfrac{1}{6} + \tfrac{1}{2})$$

Here, as always, we start with the brackets:

$$\tfrac{1}{6} + \tfrac{1}{2} = \frac{1+3}{6} = \tfrac{4}{6} = \tfrac{2}{3}$$

Next we do the multiplication:

$$\tfrac{1}{4} \times \tfrac{2}{3} = \tfrac{1}{6}$$

Finally, the subtraction:

$$\tfrac{2}{3} - \tfrac{1}{6} = \frac{4-1}{6} = \tfrac{3}{6} = \tfrac{1}{2}$$

In some ways, the line used in writing a fraction acts like a bracket in that we work out everything above it before dividing by the denominator; thus

$$\frac{4-1}{6} = \tfrac{3}{6} = \tfrac{1}{2}$$

We work out 4 − 1 before dividing by 6. The same rule applies when we meet a double-decker fraction, like

$$\frac{\tfrac{1}{2} - \tfrac{1}{3}}{\tfrac{3}{8}}$$

We first calculate $\tfrac{1}{2} - \tfrac{1}{3} = \dfrac{3-2}{6} = \tfrac{1}{6}$

Then we divide by $\tfrac{3}{8}$, $\tfrac{1}{6} \times \tfrac{8}{3} = \tfrac{4}{9}$

EXERCISE 5F

Evaluate:

1. $\tfrac{2}{5} + \tfrac{3}{4} \times \tfrac{5}{6}$
2. $(\tfrac{2}{5} + \tfrac{3}{4}) \times \tfrac{5}{6}$
3. $(\tfrac{2}{3} - \tfrac{1}{2}) - \tfrac{1}{15}$
4. $\tfrac{2}{3} - (\tfrac{1}{2} - \tfrac{1}{15})$
5. $3\tfrac{1}{7} \times (3\tfrac{1}{3} + 7\tfrac{1}{6})$
6. $2\tfrac{2}{3} - 1\tfrac{2}{5} \times (\tfrac{1}{2} + \tfrac{3}{7})$
7. $(2\tfrac{2}{3} - 1\tfrac{2}{5}) \times (\tfrac{1}{2} + \tfrac{3}{7})$
8. $\dfrac{2\tfrac{1}{4} + 3\tfrac{3}{8}}{7\tfrac{1}{2}}$
9. $\dfrac{7\tfrac{1}{5} - 3\tfrac{2}{3}}{1\tfrac{4}{5} + \tfrac{2}{5}}$
10. $\dfrac{6\tfrac{3}{4} - 3\tfrac{1}{2} \times (\tfrac{1}{3} + \tfrac{3}{7})}{(4\tfrac{2}{5} + 3\tfrac{3}{7}) \div \tfrac{18}{35}}$

Problems with Fractions

Now we shall look at a few simple problems which our knowledge of fractions will help us to solve.

(*a*) Find the cost of 3 ton 12 cwt of coke at £18·75 per ton.

3 ton 12 cwt can be written as $3\tfrac{3}{5}$ tons, and £18·75 can be written as £$18\tfrac{3}{4}$. Thus the total cost is

$$\begin{aligned} &\pounds 18\tfrac{3}{4} \times 3\tfrac{3}{5} \\ &= \pounds\tfrac{75}{4} \times \tfrac{18}{5} \\ &= \pounds\tfrac{15}{2} \times \tfrac{9}{1} \\ &= \pounds\tfrac{135}{2} = \pounds 67{\cdot}50 \end{aligned}$$

(b) Find the area of a rectangle $4\tfrac{1}{2}$ ft long and $2\tfrac{2}{3}$ ft wide.

$$\begin{aligned} \text{The area is } &4\tfrac{1}{2} \times 2\tfrac{2}{3}\ \text{ft}^2 \\ &= \tfrac{9}{2} \times \tfrac{8}{3}\ \text{ft}^2 \\ &= \tfrac{3}{1} \times \tfrac{4}{1}\ \text{ft}^2 = 12\ \text{ft}^2 \end{aligned}$$

(c) Find the perimeter of a room $10\frac{2}{3}$ ft long and $8\frac{1}{4}$ ft wide.

The perimeter is $(10\frac{2}{3} + 8\frac{1}{4} + 10\frac{2}{3} + 8\frac{1}{4})$ ft

$$= 36\frac{8+3+8+3}{12} \text{ ft}$$

$$= 36\frac{22}{12} \text{ ft}$$

$$= 37\frac{5}{6} \text{ ft}$$

EXERCISE 5G

1. Find the cost of $3\frac{1}{3}$ yd of material at $49\frac{1}{2}$p per yard.

2. If $7\frac{1}{2}$ lb of fertilizer cost £1·$12\frac{1}{2}$, how much does 1 lb cost?

3. If bacon costs 24p per lb, how many pounds can you buy for 54p?

4. Find the area of a rectangle 2 ft 4 in long and 2 ft 3 in wide, giving your answer in square feet.

5. The volume of a can is 35 in^3 and the area of the base is $4\frac{2}{3}$ in^2. Find its height.

CHAPTER 6

THE METRIC SYSTEM

IN THE eighteenth century many men on the continent of Europe felt it very inconvenient to have different units of measurement, not only in different countries, but sometimes even within one country, and during the French Revolution a new kind of measure, called a metre, was devised, based on the size of the earth. This was adopted as the standard unit in France in 1840, and nowadays most European countries use it, not only for scientific work, but for everyday measurement. It is in use in Great Britain for scientific work, and will be adopted for general use.

You must have noticed that distances for running and swimming races are usually specified in metres. The metre is rather longer than a yard, approximately 39·37 in long. Longer and shorter measures are based on it, by multiplying it and dividing it by 100 and 1000. Each of these has a separate name, and there is a standard abbreviation for each:

$$\left.\begin{array}{l}\text{1 millimetre (mm)} = \frac{1}{1000}\text{ metre}\\ \text{1 centimetre (cm)} = \frac{1}{100}\text{ metre}\end{array}\right\}\text{You should look at these on a ruler.}$$

1 metre (m)

1 kilometre (km) = 1000 metres

The equivalents in British lengths are ($\bumpeq$ means "approximately equal to"):

$$1\text{ in} \bumpeq 2\cdot54\text{ cm}$$
$$39\cdot37\text{ in} \bumpeq 1\text{ m}$$
$$1\text{ mile} \bumpeq 1\tfrac{3}{5}\text{ km}$$

Metric weights are based on the gramme, which is the weight of 1 cubic centimetre of water and larger and smaller weights use the same prefixes as those for lengths.

1 milligramme (mg) = $\frac{1}{1000}$ gramme (mostly used for scientific measurement.)

1 gramme (g)

1 kilogramme (kg) = 1000 g—about 2·2 lb, the unit most used in commerce.

1000 kg is called a tonne or metric ton, and is nearly the same weight as a ton.

One of the great conveniences of the metric system is that compound quantities (e.g. 6 lb 12 oz) are not needed. A length such as 2 m 46 cm can more easily be written 2·46 m, since 46 cm is $\frac{46}{100}$ m, and this fraction can be written in decimals as 0·46. When a French housewife buys a joint of meat which weighs 2 kg 250 g, the price-card on the meat expresses the weight as 2·25 kg.

EXERCISE 6A

Write the number of:

1. cm in 1 m.

2. m in 1 km.

3. cm in 1 km.

4. cm in 3 m.

5. g in 2 kg.

6. mg in 4 g.

Express the following lengths or weights in the units stated:

7. 2 m 4 cm in (i) m, (ii) cm.

8. 240 cm in m.

9. 3 km 40 m in (i) m, (ii) km.

10. 6·7 km in (i) m, (ii) cm.

11. 35 m in km.

12. 8 g in mg.

13. 762 g in kg.

14. If I want about a pound of sugar in a French grocery shop, what metric equivalent ought I to ask for?

15. About how far in British units is a race of 1500 m?

16. If 5 miles $\simeq$ 8 km, how many miles is it from Paris to Lyons (472 km)? How many km is it from Dieppe to Amsterdam (325 miles)?

Just as we express areas and volumes in British units as *square* inches or *cubic* yards, so in the metric system areas are expressed in units such as *square* centimetres (cm^2) or *cubic* metres (m^3). The common unit of land measurement is the *hectare* (ha), which is 10 000 m^2 (about $2\frac{1}{2}$ acres).

The common unit of capacity is the *litre*, which is the volume of one kilogramme of water. Since the cubic centimetre is the volume of one gramme of water, we have the relationship that

$$1 \text{ cm}^3 = \tfrac{1}{1000} \text{ litre}$$

For this reason a cubic centimetre is often known as a *millilitre* (ml). A litre is equal to a little over $1\frac{3}{4}$ pints.

Many countries besides Great Britain have a decimal basis for their currency.

In Switzerland 100 centimes (c) = 1 franc (fr).

In America 100 cents (c) = 1 dollar ($).

EXERCISE 6B

1. How many square millimetres are there in (i) 1 cm^2; (ii) 1m^2?

2. How many cubic centimetres are there in 1 m^3?

3. Express 1 cm^3 as a decimal fraction of 1 m^3.

4. Express 4800 cm^2 in square metres.

5. Express 340 cm^3 in litres.

6. How many grammes does 56 cm^3 of water weigh?

7. How many millilitres are there in (i) 5 litres; (ii) 0·7 litres; (iii) 0·073 litres?

8. If 1 litre $\simeq$ $1\frac{3}{4}$ pints, how many pints is (i) 3 litres; (ii) 250 ml?

9. How much would 20 m of rope cost, if 1 m cost fr 0·4?

10. Find the cost of 15 gal of petrol at $0·43 per gallon.

11. How many cubic centimetres is 0·3 litres? x litres?

12. 10 m of flex weighs 12 g. Find the weight of 3 km of wire in kg.

13. If the average length of a runner's pace is 1·32 m, how far will he travel in 82 paces?

14. An American farm hand is paid 84 cents an hour. How much (in \$) will he earn in a 40-hour week?

15. If 475 ml of wine is poured from a full bottle ($\frac{3}{4}$ litre) how much remains?

16. 1 kg $\simeq$ 2·2 lb (or 5 kg $\simeq$ 11 lb). A man weighs 15 stone 10 lb. What is his weight in kg?

17. If oil is about four-fifths the weight of an equal volume of water, what is the weight of 1 litre of oil?

18. A storage tank is 5 m long, 3 m wide, and 1 m deep. How many cubic metres will it hold? How many litres? If it is filled with water, what is the weight of the water in kg? (1000 litres = 1m^3.)

19. An empty plastic container weighs 200 g. If 0·6 litres of water are poured into it, find the total weight of water and container.

20. What is the perimeter of a rectangular field, in km, which is 426 m long and 317 m wide? What is its area, in hectares, correct to one place of decimals?

Significant Figures

If we measure the length of a line and find it to be 325 cm long (to the nearest centimetre) we can also write this length as:

0·003 25 km
or 3·25 m
or 3250· mm

all correct to the same degree of accuracy.

The digits 325 occur whichever way we write the length and are the *significant figures* of this measurement. The noughts before the 325 in 0·003 25 km, and that after the 5 in 3250 mm, since this measurement is correct to the nearest cm, are not significant. Significant figures are those which must be kept wherever the decimal point is, and they are counted from left to right, beginning with the first digit which is not zero.

The number of significant figures given in a length is important in deciding what degree of accuracy has been used in measuring it. Suppose a carpenter measures a length of wood as 54 cm, *correct to the nearest mm*. To how many significant figures is this correct?

54 cm = 540 mm

Since it is to the nearest mm, the limits of his possible error are from 539·5 mm to 540·5 mm. (Had the true length been, say, 539·4 mm, it would

become 539 mm, corrected to the nearest mm, and had it been 540·6 mm it would have become 541 mm when taken to the nearest mm.) So the length is correct to *three* significant figures and can be written

540 mm
or 54·0 cm

(notice that the zero *is* significant in both cases).

Example:

Express 30·476 correct to 3 significant figures.

Taking the first three figures we have 30·4, but the fourth figure is 7, and 30·47 is nearer to 30·5 than to 30·4.

$\therefore$ the value is 30·5 correct to 3 significant figures.

(*Note:* This is often abbreviated to "correct to 3 figures.")

Example:

Express 0·004 073 correct to 3 figures.

The first significant figure is 4. Taking the first three, we have 0·004 07. The next figure is below 5, so that 0·004 073 is nearer to 0·004 07 than to 0·004 08.

$\therefore$ the value is 0·004 07 correct to 3 figures.

Example:

Express 52 700 correct to 2 figures.

The first significant figures are 52. The next figure is above 5, and 52 700 is nearer to 53 000 than to 52 000.

$\therefore$ The value is 53 000 correct to 2 figures.

Notice that the value, when corrected is *not* 53. The basic *size* of the number is not altered.

The important thing to remember is that we should not give answers which are apparently more accurate than the measurements we began with.

Suppose we measure a book with a ruler and find that it is 9·7 in long and 8·1 in wide, correct to the nearest $\frac{1}{10}$ of an inch (or to 2 figures). The area, according to these measurements should be 9·7 × 8·1 or 78·57 in². But the length is only accurate to $\frac{1}{10}$ of an inch, so might range from 9·65 to 9·75 in.

Similarly, the width might range between 8·05 to 8·15 in. The largest possible area would then be 9·75 × 8·15, or 79·4625 in², and the smallest possible area would be 9·65 × 8·05, or 77·6825 in². The possible areas range from just under $79\frac{1}{2}$ to just over $77\frac{1}{2}$ in². The area cannot therefore, be given to the nearest square inch, but only to *one* significant figure. The best we can do is to say that the area is 80 in², to one significant figure.

Example:

Two lengths are given as 5·6 in and 3·4 in, both correct to 2 significant figures. Between what limits does their sum lie?

The greatest possible lengths are 5·65 in and 3·45 in (total 9·1 in).

The least possible lengths are 5·55 in and 3·35 in (total 8·9 in).

∴ the greatest and least values are 9·1 and 8·9 in.

EXERCISE 6C

Write the following quantities correct to the number of significant figures given in brackets:

1. 1·764 m (2)

2. 506 km (2)

3. 0·0471 mm (1)

4. 0·7142 kg (3)

5. 0·547 m (1)

6. 93 000 miles (1)

7. 64·25 cm (2)

8. 0·004 87 m (2)

9. 3·007 litres (3)

10. 5·008 g (2)

11. 200·46 inches (3)

12. 3477 yd (2)

13. A car's average speed is said to be 73·4 mile/h correct to 3 figures. Between what distances might it travel in 1 hour? In 3 hours?

14. The sides of a triangle are measured as 3·45 in, 2·17 in and 3·04 in, each correct to 3 figures. What is the greatest and least possible measurement of the perimeter of the triangle?

15. (i) A man paid £2300 in cash for a car.

(ii) He owns a farm of about 2300 acres.

To how many significant figures are each of these statements likely to be correct?

16. A runner's average stride is 1·6 m long, correct to 2 significant figures. What is the greatest and least possible distance for each stride? For 190 strides?

17. If each of the amounts 21p, 29p, 13p is correct to the nearest penny, between what limits does their sum lie?

18. If a table is 1·7 m long and 1·2 m wide, each correct to 2 figures, what is the area, and how accurate can the answer be?

19. A rectangle has sides 4·2 cm, 3·4 cm, correct to the nearest mm. Find the area in cm^2.

20. If there are 14 200 000 people of working age and 2·65 per cent are unemployed (both correct to 3 figures), how many unemployed are there?

Changing Quantities into Decimal Fractions

While Imperial weights and measures remain in use it is useful to know how to change them into decimals.

Example:

Express 1 mile 3 furlongs 6 chains as a decimal of 1 mile, to 2 places.

1. Change the chains to furlongs (10 chains = 1 furlong):

$$10\overline{)6} \text{ chains}$$
$$0{\cdot}6 \text{ furlongs}$$

2. Add the furlongs and divide by 8 to obtain miles:

$$8\overline{)3{\cdot}6} \text{ furlongs}$$
$$0{\cdot}45 \text{ miles}$$

3. Add the miles (no correction is necessary here):

$$1{\cdot}45 \text{ miles.}$$

Example:

Express 2 quarters 5 lb 8 oz as a decimal of 1 cwt, to 3 places.

Divide by 16, to change to lb:

$$16\overline{)8} \text{ oz}$$
$$0{\cdot}5 \text{ lb}$$

Add the lb and divide by 28 (or by7 and 4):

$$28\overline{)5{\cdot}5} \text{ lb} \qquad 7\overline{)5{\cdot}5} \text{ lb}$$
$$0{\cdot}1964 \text{ quarters} \qquad 4\overline{)0{\cdot}7857}$$
$$0{\cdot}1964 \text{ quarters}$$

Add the quarters and divide by 4:

$$4\overline{)2{\cdot}1964} \text{ quarters}$$
$$0{\cdot}5491 \text{ cwt}$$

$\therefore$ 2 quarters 5 lb 8 oz = 0·549 cwt to 3 places.

EXERCISE 6D

In No. 1-10 give the first quantity as a decimal of the second. If they do not give exact answers, work out 4 places of decimals and correct to 3.

1. 4 ft, 7 yd.
2. 7 hours, 24 hours.
3. 3 oz, 1 lb.
4. 14 in, 1 yd.
5. 5 oz, 1 lb.
6. 16 in, 1 yd.
7. 15 cm, 1 m.
8. 15 hours, 24 hours.
9. 25 ft, 1 furlong.
10. 2 quarts, 3 gal.

In No. 11-18, express as decimals of the unit in brackets, giving the answer correct to 3 places of decimals.

11. 1 ton 11 cwt 2 quarters (ton)
12. 4 h 13 min 42 s (hour).
13. 3 gal 1 quart 2 pt (gallon).
14. 2 miles 5 furlongs 45 yd (miles).
15. 3 yd 2 ft 10 in (yard).
16. 1 quarter 15 lb 2 oz (quarter)
17. 4 miles 564 yd (mile).
18. 2 tons 2 cwt 84 lb (ton).

CHAPTER 7

RATIOS

SUPPOSE we want to make a scale drawing of a sideboard. We might decide to make our drawing 6 in long. It obviously won't do to reason like this: "My drawing is 6 in long, the actual sideboard is 6 ft long, therefore my drawing is 5 ft 6 in less than the sideboard, so I must make the height of my drawing 5 ft 6 in less than the real height."

It would be impossible to make a scale drawing if we reasoned like this! We all use scale drawings for models and for maps, and as you know, what we do is to *compare* the real length and the drawn length in the form of a fraction.

Real length 6 ft
Drawn length 6 in

$\therefore$ The drawn length is $\frac{1}{12}$ of the real length.

$\therefore$ The drawn height must also be $\frac{1}{12}$ of the real height,

$\frac{1}{12} \times 30$ in, or $2\frac{1}{2}$ in.

Notice how we arrive at this fraction, $\frac{1}{12}$.

It is $\dfrac{6 \text{ in}}{6 \text{ ft}}$, or $\dfrac{6 \text{ in}}{72 \text{ in}}$; the drawn length divided by the real length

$= \frac{1}{12}$

When we use a map on the scale of 1 in to 1 mile, every inch on the map represents 1 mile (or 63 360 in) on the ground. So the map is $\frac{1}{63\,360}$ of the size of the real thing.

Quantities are often compared by saying what fraction one is of the other. In the case of the sideboard, each measurement on the drawing is $\frac{1}{12}$ of that on the real thing. This fraction is called the *ratio* between them, and is written 1 : 12. The ratio of the distance on the map to the distance on the ground is written as 1 : 63,360.

$\frac{2}{7}$ as a fraction is 2 : 7 as a ratio.
$\frac{5}{11}$ as a fraction is 5 : 11 as a ratio.
$\dfrac{x}{y}$ as a fraction is $x : y$ as a ratio.

Perhaps it is just as well to remember what we mean when we say that one quantity is $\frac{2}{7}$ of another. What we are saying is that if we divide the smaller quantity into 2 parts, the larger will be equal to 7 of those same parts. Ratio form, 2 : 7, is another way of saying this.

Example:

Express the ratio 30p to £1·80 in its simplest form.

$$\frac{30\text{p}}{£1{\cdot}80} = \frac{30\text{p}}{180\text{p}} = \tfrac{1}{6} \text{ or } 1:6$$

(30p is $\frac{1}{6} \times$ £1·80, or £1·80 is $\frac{6}{1} \times$ 30p)

Notice that (i) ratios, like fractions, are made as simple as possible. (ii) When one quantity is expressed as a ratio of another they must be in the same units.

Example:

Express 2 cm to 3 m in its simplest form.

$$\frac{2\text{ cm}}{3\text{ m}} = \frac{2\text{ cm}}{300\text{ cm}} = \tfrac{1}{150} \text{ or } 1:150$$

Example:

Express the ratio $3\frac{1}{4}$ hours to 20 minutes in its simplest form.

$$\frac{3\frac{1}{4}\text{ h}}{20\text{ min}} = \frac{195\text{ min}}{20\text{ min}} = \tfrac{39}{4} = 39:4$$

Example:

Express the ratio $1\frac{1}{2}$ pt to 2 gal in its simplest form.

$$\frac{1\frac{1}{2}\text{ pt}}{2\text{ gal}} = \frac{1\frac{1}{2}\text{ pt}}{16\text{ pt}} = \tfrac{3}{32} = 3:32$$

Example:

Express the ratio $4\frac{1}{2}$ lb to 14 oz in its simplest form.

$$\frac{4\frac{1}{2}\text{ lb}}{14\text{ oz}} = \frac{72\text{ oz}}{14\text{ oz}} = \tfrac{36}{7} = 36:7$$

EXERCISE 7A

Simplify the following ratios, expressing them as fractions in their lowest terms:

1. 6p : 30p
2. 2 ft : 3 yd
3. 220 yd : 1 mile
4. 1 kg : 50 g
5. 0·6 : 1
6. $2\frac{1}{2}$ pt : 1 gal
7. 2 tons : 6 cwt
8. 30 cm : 1 m
9. £1·56 : 39p
10. 17 : 85

Conversion to Whole Numbers

Ratios are usually expressed as whole numbers, so that if we are comparing fractions or decimals we change them to whole numbers.

Example:

Express the ratio $\frac{5}{6} : \frac{3}{4}$ in its simplest form.

$$\frac{5}{6} : \frac{3}{4} = \frac{\frac{5}{6}}{\frac{3}{4}} = \frac{5}{6} \times \frac{4}{3}$$
$$= \frac{10}{9} \text{ or } 10 : 9$$

Example:

Express the ratio 3·25 : 1·25 in its simplest form.

$$\frac{3{\cdot}25}{1{\cdot}25} = \frac{325}{125}$$
$$= \frac{13}{5} \text{ or } 13 : 5$$

EXERCISE 7B

Simplify the following ratios (remember that when you construct the ratio, the quantities must be in the same units):

1. 20 : 2·4
2. 3·5 : 0·85
3. $3\frac{1}{4} : 2\frac{1}{5}$
4. 0·7 : 13·3
5. $1\frac{2}{3} : 5\frac{5}{11}$
6. 90 ft/s : 20 yd/min
7. 84 mile/h : 88 ft/s
8. 200 cm : 1 km
9. 300 ml : 2 litres
10. 7 lb : 2 cwt

11. A shop has two tables, one 3 ft square, the other 4 ft square. What are the ratios of the lengths of the sides? Of their areas?

12. A factory employs 280 men and 42 women. What is the ratio of the number of men to the number of women? Of the number of women to the number of men? What is the ratio of the number of women to the total number employed there?

13. One rug is 150 cm long and 70 cm wide. Another is 200 cm long and 84 cm wide. What is the ratio of the area of the first to that of the second?

14. A map is drawn on the scale of 12 in to 1 mile. Find the ratio of a distance on the map to the distance it represents on the ground.

15. What is the ratio between 1 mile and 1 nautical mile (6 080 ft)?

Finding Quantities from Given Ratios

If we are given a ratio between two quantities we can always find one if given the other.

The ratio between the lengths of these lines, AB and CD, is 3 : 4. Suppose we are now told that the length of AB is 21 cm. This consists of 3 parts as against the 4 of CD. If 3 parts are 21 cm, 1 part is 7 cm and CD, which is 4 of the same parts, must be 28 cm long.

Example:

The ratio between the price of a car when new, and its second-hand price after 1 year is 17 : 13. What will its second-hand price be if it cost £850 new?

£850 represents 17 parts
$\therefore$ 1 part will be $\frac{850}{17}$ or £50
$\therefore$ 13 parts will be $13 \times £50 = £650$.

EXERCISE 7C

1. The ratio between lengths on a plan of a house and the house itself is 1 : 96. What length, in feet, on the house is represented by $2\frac{1}{2}$ in on the plan?

2. The top and bottom marks in a class are in the ratio 9 : 2. If the top boy scores 117 marks, what is the mark of the boy at the bottom?

3. The old lamp-posts in a street were replaced by new ones, which were higher in the ratio 8 : 5. If the old lamp-posts were 12 ft 6 in high, how tall are the new ones?

4. A bank manager earns £1,200 per annum, and his savings are in the ratio 3 : 20 to his income. How much does he save every year?

5. The cost of wool for embroidering a firescreen is 65p. If more expensive silk thread is used, the ratio of the price being 7 : 5, how much will the silk cost?

6. The time taken for a journey from London to Glasgow by train and by aircraft, respectively, is in the ratio 10 : 3. How long will the plane take if the train takes 8 hours? How long will the train take if the plane takes $3\frac{1}{2}$ hours?

7. The ratio of 1 mile to 1 km is approximately 8 : 5. What is 3 miles in km? 10 km in miles?

8. The volumes of two cubes are in the ratio 27 : 125. If the length of the side of the smaller is 6 cm, find the length of the larger. (*Hint:* Find the ratio of the lengths of the sides first.)

Division in a Given Ratio

Suppose a farmer decides to share his farm among his three sons. He has 240 acres and gives James, his eldest son, 100 acres; John, his second son, 80 acres; and Mark, his youngest son, 60 acres. We could compare the acreage each received by saying that:

(1) The ratio of James's land to John's is 100 : 80, or 5 : 4

(2) The ratio of John's land to Mark's is 80 : 60, or 4 : 3

and (3) The ratio of James's land to Mark's is 100 : 60, or 5 : 3

A convenient way of comparing the areas they received is to say that the farmer divided his land in the ratio of

5	:	4	:	3
(James)		(John)		(Mark)

This compresses into one statement (1), (2) and (3) above. The farmer has divided his land into 12 (i.e. $5 + 4 + 3$) parts; each part being one-twelfth of 240 acres, which is equivalent to 20 acres. Five parts (5×20—100 acres) are given to James; 4 parts (4×20—80 acres) to John; and 3 parts (3×20—60 acres) to Mark.

Thus, if we have an expression which states

$$x : y : z = 3 : 5 : 7$$

It means

$$\begin{aligned} &(1)\ x : y = 3 : 5 \\ &(2)\ y : z = 5 : 7 \\ \textit{and}\ &(3)\ x : z = 3 : 7 \end{aligned}$$

EXERCISE 7D

1. Divide £32 in the ratio 5 : 3.

2. Divide 3 kg in the ratio 5 : 7.

3. Divide 5 hours in the ratio 1 : 2 : 3.

4. Divide 70 yd in the ratio 3 : 5.

5. Find three whole numbers in the ratio $1\frac{1}{6} : 2\frac{1}{3} : 3\frac{1}{2}$. (*Hint:* Change the terms of the ratio to fractions with a common denominator.)

6. Draw a line AB 4·5 in long. Point X on AB divides the line in the ratio 2 : 7. Find the length of AX by drawing and measurement and XB by calculation. (Their sum should be 4·5 in.)

7. A line 7·8 cm long is divided in the ratio 4 : 9. Find the length of the shorter part by drawing and measurement.

8. Share £1·50 between 3 people in the ratio 6 : 3 : 1.

9. Three men hire a car for a day. The charge is £4·50. If Jones drove it for 3 hours, Smith for 2 hours and Robinson for 4 hours, how should they divide the cost?

10. Find 3 whole numbers in the ratio of 0·2 km : 105 m : 50 cm.

11. An astronaut during orbit, reads his instruments, controls his craft, and reports to base in the time ratio 4 : 5 : 1. How much time does he spend reading his instruments in one 90-minute orbit of the earth?

12. Black, Brown and Green are scaffolders. Black can erect a scaffolding in 4 hours, Brown in 5 hours and Green takes 6 hours. If they put up the scaffolding together and are paid £5·55 between them, how should they share the money? (*Hint:* How much does each do in *one* hour?)

13. In a mathematics test, the first question carries twice as many marks as the second, and the second 3 times as many as the third. The total for the test is 30 marks.

How many marks can be obtained for the first question?

14. A car averages 30 mile/h between two towns. A train averages 50 mile/h between the same places. Find the ratios of the times taken by the car to that of the train. What is the ratio of their speeds?

Proportional Parts

Suppose that we were told that the length : width ratio of a room is 4 : 3 and the width : height ratio is 5 : 4. The width occurs in both ratios and we could write:

length	:	width	:	height
4	:	3		
		5	:	4

How can we connect the length and the height? By making the width a term common to both ratios.

If we change the width to 15 (a multiple of 3 and 5) this will mean multiplying the width term in the first ratio by 5 and so the length term must also be multiplied by 5 if the ratio is not to be altered. Similarly, in the second

ratio, both terms must be multiplied by 3, and our pair of ratios now look like this:

length	:	width	:	height
20	:	15		
		15	:	12

The ratios can now be combined and

$$\text{length} : \text{width} : \text{height} = 20 : 15 : 12$$

(This means, too, that we can now compare the length and the height by the ratio 20 : 12, or 5 : 3.)

EXERCISE 7E

1. If $x : y = 3 : 2$ and $y : z = 3 : 5$, find the ratios $x : y : z$ if x, y and z are whole numbers.

2. If Smith's salary is greater than Jones's in the ratio 5 : 4, and Jones's is greater than Robinson's in the ratio 5 : 3, what is the ratio between Smith's salary and Robinson's?

3. Find the ratio between a and c when $a : b = 8 : 9$, and $b : c = 8 : 11$.

4. The perimeter of a triangle is 5·6 in, and the lengths of the sides are in the ratio 2 : 5 : 7. What is the sum of the shortest and longest sides?

5. A photographer takes a print from a negative. The length of the sides of the print are enlarged by comparison with the negative in the ratio 5 : 3. If the print is afterwards reproduced in a book, and the lengths of its sides reduced in the ratio 7 : 9, how does the length of the sides of the reproduction compare with that of the negative?

Increases or Decreases in a Given Ratio

If, during Sales Week, the price of a pair of shoes is reduced from £3 to £2, the ratio of the *new* price to the original is 2 : 3 or $\frac{2}{3}$. On the other hand, if manufacturing costs go up, and the shopkeeper has to increase the price from £3 to £4, the ratio of the new price to the original is 4 : 3 or $\frac{4}{3}$.

We could say:

Sale

Decrease from £3 to £2.
Decrease in ratio 2 : 3 or $\frac{2}{3}$.
(New price is $\frac{2}{3}$ of old.)

Price Increase

Increase from £3 to £4.
Increase in ratio 4 : 3 or $\frac{4}{3}$.
(New price is $\frac{4}{3}$ of old.)

It is clear that if we knew only the original price and the ratio of increase or decrease, we could find the new price:

Sale

Original price £3
Decrease in ratio 2 : 3 or $\frac{2}{3}$
$\therefore$ New price $= £3 \times \frac{2}{3}$
$= £2$

Price Increase

Original price £3
Increase in ratio 4 : 3 or $\frac{4}{3}$
$\therefore$ New price $= £3 \times \frac{4}{3}$
$= £4$

The fractions $\frac{2}{3}$ and $\frac{4}{3}$, obtained from the ratios, are called MULTIPLYING FACTORS. Notice that the new quantity is always the *numerator* of the fraction, and the original quantity is the *denominator*. Using ratios in this way there are three things we can do:

1. Given the original quantity and the ratio we can find the new quantity.
2. Given the new quantity and the ratio we can find the original quantity.
3. Given the new and the old quantity we can find the ratio and the multiplying factor.

Example:

Increase 56p in the ratio 8 : 7.

The multiplying factor is $\frac{8}{7}$ $\left(\text{remember the general rule that } x : y = \frac{x}{y}\right)$

$\therefore$ New value $= 56\text{p} \times \frac{8}{7}$
$= 64\text{p}$

Example:

Decrease 2 ft 8 in in the ratio 3 : 4.

Decreased length $= 32 \text{ in} \times \frac{3}{4}$
$= 24$ in
$= 2$ ft

Example:

If the time of a journey by train is reduced in the ratio 5 : 9 and the new scheduled time is 1 hour 15 minutes what was the original schedule time?

The new schedule takes 5 units of time as against 9 of the old.

But the new time is 75 minutes
$\therefore$ The old must be $75 \times \frac{9}{5}$ minutes
$= 135$ minutes
$= 2$ hours 15 minutes

The multiplying factor here might seem, at first sight, to be reversed. But the *known* time, i.e. 75 minutes, is in this question, the quantity we start from. We could re-write the question to read: "The present journey takes 75 minutes, and the previous schedule took more time in the ratio 9 : 5." Or, keeping the question in its first form, we could solve the question by an equation:

Let x minutes be the original time.

$$\text{Multiplying factor} = \frac{\text{new quantity}}{\text{original quantity}}$$
$$= \tfrac{5}{9}$$
$$\text{Then } x \times \tfrac{5}{9} = 75$$
$$\therefore 5x = 75 \times 9$$
$$\therefore x = \frac{75 \times 9}{5}$$
$$= 135$$

$\therefore$ Old time was 2 hours 15 minutes.

Example:

By what ratio must 2 ft 3 in be increased to become 6 ft?

The ratio 6 ft : 2 ft 3 in
= 72 : 27 (Changing the length to inches)
= 8 : 3 (Dividing each side by 9)

Hence the multiplying factor which will change 2 ft 3 in to 6 ft is $\frac{8}{3}$ (and the multiplying factor which will change 6 ft to 2 ft 3 in is $\frac{3}{8}$).

EXERCISE 7F

In No. 1-6 increase these quantities in the given ratio:

1. 20; 5 : 4
2. 2 kg; 3 : 2
3. 4 ft; 9 : 8
4. 3 hours; 7 : 6
5. 2 cm; 50 : 1
6. 12 mile/h; 5 : 3

In No. 7-12 decrease these quantities in the given ratio:

7. 22 lb; 5 : 11
8. 45 yd; 2 : 9
9. 32 ft/s; 5 : 8
10. 260 g; 5 : 13
11. 51p; 16 : 17
12. £1·90; 2 : 19

In No. 13-18 increase or decrease these quantities in the given ratio:

13. 25 lb; 6 : 5
14. 49p; 15 : 14
15. 1 litre; 3 : 10
16. 2 gal; 3 : 4
17. 51 minutes; 10 : 17
18. 37·4m; 13 : 11

19. A girl has her salary increased in the ratio 5 : 4 and then earns £850 per annum. What was her salary before her increase?

20. The weights of 2 jars of honey are in the ratio 13 : 8. If the heavier weighs 2 lb 7 oz, what does the lighter weigh?

21. The price of petrol goes up in the ratio 10 : 9. If the new price is 35p what was it before the increase?

22. A boy scores 18 out of 20 in one test, and 15 out of 20 in the next. In what ratio has his mark decreased?

23. A room is 24 ft long. In what ratio is the length reduced if it appears on an architect's plan 3 in long?

24. A shirt which normally costs £2·25 is sold in a sale for £1·80. In what ratio is the price reduced?

25. A painting, size 45 cm by 36 cm, is reproduced in a book. The sides are reduced in the ratio 2 : 3. What are the lengths of the sides of the reproduction? In what ratio is the area of the picture reduced?

Simultaneous Changes of Ratio

Example:

Mrs. Jones, who is rather fat, decides to cut down her consumption of chocolate biscuits by a quarter. At the same time the price of biscuits unfortunately goes up by a quarter. If she spent 40p a week on biscuits formerly, how much does she now spend?

There are two changes here:

"Cutting her consumption by a quarter" means that for every 4 pounds or ounces she used to eat, she now eats only 3.

$\therefore$ Her consumption is reduced in the ratio 3 : 4.

$\therefore$ The old price is multiplied by the factor $\frac{3}{4}$.

"The price goes up by a quarter" means that for every 4p she used to pay, she now has to pay 5p.

$\therefore$ The price increases in the ratio 5 : 4.

$\therefore$ The old price is multiplied by the factor $\frac{5}{4}$.

Since both changes occur at the same time the new price is

$$40\text{p} \times \tfrac{3}{4} \times \tfrac{5}{4}$$
$$= \tfrac{75}{2}\text{p}$$
$$= 37\tfrac{1}{2}\text{p}$$

EXERCISE 7G

1. The price of coal went up in the ratio 11 : 10. Because of this a family cut their consumption of coal in the ratio 5 : 6. What is their new monthly coal bill if it was formerly £9·60?

2. The price of tobacco was increased in the ratio 9 : 8. Mr. X, a retired mathematics master, who spent 60p a week on tobacco, decided that he would reduce his consumption each week so that he spent only the same amount as before. In what ratio will he have to reduce the amount that he buys?

3. After buying a new and larger car a man finds his petrol consumption has gone up by $\frac{3}{10}$. In what ratio has it increased?

4. When a new hydro-electric power station came into operation the price of electricity fell from 4p to 3p a unit. A family thereupon bought new electric fires and a hair-dryer, and their consumption rose from 200 to 300 units a quarter. In what ratio did their bill for electricity alter?

5. A salesman earned £110 in January and £95 in February. In what ratio have his earnings decreased? If in March he earned £105, in what ratio have his earnings changed, by comparison with his average for January and February?

Ratio Comparison

If McKenzie has 45p a week for pocket-money and saves 10p, and McKay has 90p and saves $17\frac{1}{2}$p, for which is the ratio of savings to pocket-money the greater?

McKenzie saves 10p from a total of 45p, a ratio of $\frac{10}{45}$ or 2 : 9.

McKay saves $17\frac{1}{2}$p from a total of 90p, a ratio of $\frac{35}{180}$ or 7 : 36.

A difficulty may arise in trying to compare two ratios, but if we can change both into decimal fractions with denominator of 1, we *can* compare them.

McKenzie

$$2 : 9 = \frac{2}{9} = \frac{2 \div 9}{1} \quad \text{(Dividing both top and bottom by 9)} \qquad 9\overline{)2{\cdot}0} \atop 0{\cdot}\dot{2}$$

$$= \frac{0{\cdot}\dot{2}}{1} \text{ or } 0{\cdot}\dot{2} : 1$$

McKay

$$7 : 36 = \frac{7}{36} = \frac{7 \div 36}{1} \qquad 36\overline{)7{\cdot}0} \atop 0{\cdot}194$$

$$= \frac{0{\cdot}19\dot{4}}{1} \text{ or } 0{\cdot}19\dot{4} : 1$$

Thus the ratio for McKenzie of saving to pocket money is the greater. All ratios can be expressed in the form $n : 1$ or $1 : n$. The general rule is:

(a) to convert $x : y$ or $\frac{x}{y}$ into the form $n : 1$, divide x by y and put the result in place of n.

(b) to convert $x : y$ or $\frac{x}{y}$ into the form $1 : n$, divide y by x and put the result in place of n.

EXERCISE 7H

Give the results correct to 3 significant figures where necessary.

In No. 1-6, express the ratios in the form $1 : n$.

1. 2 : 3
2. 3 : 4
3. 3 : 5
4. 8 : 3
5. 12 : 7
6. 4 : 19

In No. 7-12 express the ratios in the form $n : 1$.

7. 2 : 3
8. 3 : 4
9. 8 : 3
10. 110 : 9
11. 9 : 110
12. 35 : 27

In No. 13-16 say which is the greater of each pair of ratios.

13. 3 : 4; 4 : 5
14. 11 : 7; 19 : 12
15. 0·428571 : 1; 3 : 7
16. 29 : 7; 45 : 11

Maps and Scale Drawing

Some Ordnance Survey maps are on the scale of 1 in to 1 mile. This means that 1 in on the map represents 1 mile on the ground, which can be represented in the form of a ratio:

$$1 \text{ in} : 1 \text{ mile}$$

$$\text{or, in inches, } 1 : 63\,360 \text{ or } \frac{1}{63\,360}$$

When the scale of a map or plan is represented in the form $1 : n$ it is called the REPRESENTATIVE FRACTION (abbreviation R.F.). A house plan with an R.F. of $\frac{1}{96}$ means that 1 in on the plan represents 96 in on the house (a scale of 1 in to 8 ft).

Example:

What is the R.F. of a map on the scale of 4 cm to 1 km?

$$4 \text{ cm} : 1 \text{ km} = \frac{4 \text{ cm}}{1 \text{ km}} = \frac{4 \text{ cm}}{100\,000 \text{ cm}}$$

$$\therefore \text{R.F.} = \frac{1}{25\,000}$$

Example:

The R.F. of a map is $\frac{1}{10\,000}$. What distance on the ground is represented by 35 cm on the map?

The distance on the ground is 10 000 times greater than that on the map,

$$\therefore \text{ Distance on ground} = 35 \times 10\,000 \text{ cm}$$
$$= 3{\cdot}5 \text{ km}$$

Example:

If the R.F. of a map is $\frac{1}{40\,000}$, what distance on the map represents 3·5 miles on the ground? Give the answer in inches, correct to 2 decimal places.

The distance on the map is $\frac{1}{40\,000}$ of that on the ground,

$$= \frac{3{\cdot}5 \text{ miles}}{40\,000}$$
$$= \frac{3{\cdot}5 \times 63\,360 \text{ in}}{40\,000} = \frac{3{\cdot}5 \times 63\,36\cancel{0}}{40\,00\cancel{0}} = \frac{22\,176}{4000}$$
$$= 5{\cdot}54 \text{ in (correct to 2 decimal places)}$$

$$\begin{array}{r} 6336 \\ \times\ 35 \\ \hline 31680 \\ 190080 \\ \hline 221760 \end{array}$$

$$22176 \div 1000 = 22{\cdot}176$$

$$4\overline{)22{\cdot}176}$$
$$5{\cdot}544$$

EXERCISE 7J

In No. 1-6, find the R.F. of maps on the following scales:

1. 1 yd to 1 mile.
2. 2 in to 1 mile.
3. 10 in to 1 mile.
4. 1 in to 2 miles.
5. 1 cm to 1 km.
6. 3 in to 50 ft.

7. It is 240 miles from Glasgow to Sheffield. What would this be in inches on a map whose R.F. is $\frac{1}{720\,000}$?

8. A map of Kent has an R.F. of $\frac{1}{380\,000}$. How many miles does 2 in on map represent?

9. A map of the Netherlands has an R.F. of $\frac{1}{1\,200\,000}$. What is the distance in km between Amsterdam and the Hook of Holland, which are 6·05 cm apart on the map?

10. A map is drawn on a scale of 1 in to 10 miles. How many acres are represented by 1 in^2.

Proportion by Ratio

We have already solved some problems on proportion by the unitary method in Chapter 3. The first example given asked: "What is the cost of 12 gal of paraffin if 18 gal cost £1·80?"

Using the idea of ratio gives us a much more direct method of solving such problems:

The amount of paraffin is decreased in the ratio $\frac{12}{18}$

$\therefore$ The cost decreases in the same ratio

$\therefore$ The cost $= £1{\cdot}80 \times \frac{12}{18}$

$= £1{\cdot}20$

(This, you remember, is an example of DIRECT PROPORTION. As the amount increases, so does the cost.)

Here is the second example worked by the ratio method:

"If I can buy 5 ball-point pens for 50p, how many pens can I buy for 90p?"

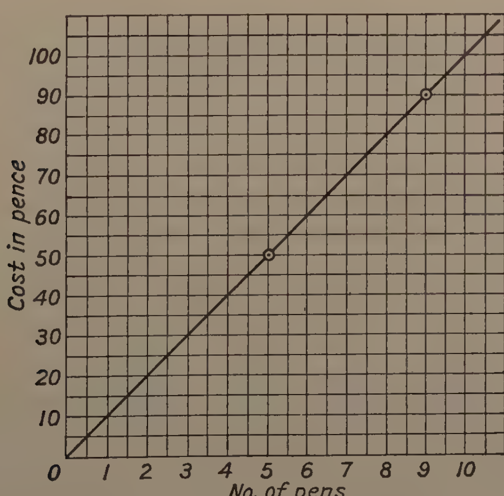
The cost increases in the ratio $\frac{90}{50}$

$\therefore$ The number of pens increases in the same ratio

$\therefore$ The number of pens is $5 \times \frac{90}{50}$

$= 9$ pens

This is also direct proportion. As the cost increases, so does the number of pens. We could plot on a graph the number of pens which could be bought for varying amounts, thus:



We know 5 pens cost 50p, 9 pens cost 90p, and no money will not buy any.

The graph is a straight line, passing through the origin. All graphs drawn to show the relationship of quantities in *direct* proportion are of this type. If you have not already done so, you should construct some graphs of direct proportion to use as Ready Reckoners (km to miles, and kg to lb, would be useful ones).

Now look back at the worked example in Chapter 3, page 35. The question, briefly, was: "How long would a train take to do a journey at 30 mile/h if it took 10 hours at 39 mile/h.

This is inverse proportion. At *less* speed the train takes *longer*. The time increases in the ratio 39 : 30 or $\frac{39}{30}$.

At 39 mile/h the journey takes 10 hours
At 30 mile/h the journey takes $10 \times \frac{39}{30}$
$= 13$ hours

Example:

A book is printed with 34 lines to a page. It has 260 pages. How many pages would be required if a larger type is used which only permits 30 lines to a page?

Quite evidently the book will need more pages.
The number of lines is decreased in the ratio 30 : 34 or $\frac{30}{34}$
$\therefore$ The number of pages will increase in the ratio 34 : 30 or $\frac{34}{30}$

$\therefore$ The new number of pages required will be $\dfrac{26\cancel{0} \times 34}{3\cancel{0}} = \dfrac{884}{3}$
$= 294\frac{2}{3}$

$\therefore$ 295 pages will be necessary

$$\begin{array}{r} 26 \\ 34 \\ \hline 104 \\ 780 \\ 3\overline{)884} \\ \hline 294 \end{array}$$

EXERCISE 7K

Here are examples of both direct and inverse proportion. You must decide which is which, and solve them by the ratio method.

1. An oil heater uses 3 pt of oil in 4 hours. How long would 1 gal last?

2. 4 in on a map represents 15 miles on the ground. What would 9 in represent?

3. Three machines bottle 1000 gal of milk in 56 minutes. If one machine broke down, how long would the remaining machines take to bottle the same quantity?

4. A chicken farmer cut off by snow has enough feed for 400 hens for 5 days. If he discovers that 80 of the hens have died from the cold, how long will the feed now last?

5. A man pays £90 a year in rates on a house assessed at £108. If he moves to a bigger house in the same district, assessed at £120, what will he now have to pay in rates?

6. A record which should take 13 minutes to play at 45 rev/min is put on a record-player and played by mistake at 78 rev/min. Assuming you could stand the noise, how long would it now take to play?

7. An employer pays out £425 a week in wages. The average hourly rate of his workers is 34p per hour. If their wage is increased by 2p an hour, what will be his new weekly wages bill?

8. If I can now buy 8 lb of jam for £1·40, how much will I be able to buy for the same money if the price increases in the ratio 16 : 15?

9. If x men can paint a house in y days, how long will z men take if they work at the same rate?

10. If 35 cm of tubular scaffolding weighs 1·5 kg how much would 154 cm of similar tubing weigh ?

11. An enclosure 510 ft long will hold 4000 people; an adjoining enclosure of the same width is 710 ft long. How many people will it hold?

12. Two cogwheels mesh in a gearbox. The smaller, which has 16 teeth is revolving at 3762 rev/min. If the larger has 22 teeth how fast will it be revolving?

13. On a map whose scale is 1 in to 20 miles, the distance between two towns is 3·35 in. What will be the distance between the same two towns on a map on the scale of 1 in to 67 miles?

14. If the odometer (distance recorder) of a car is accurate with normal tyres whose outer circumference is 84 in, and heavy-duty tyres are fitted whose circumference is 88 in, will the odometer record more miles than it should, or less? What will it record when the car has, in fact, covered 143 miles?

15. If an ironmonger makes a profit of 25p in the £1 on the mowers he sells, how much does he pay for a motor mower selling for £32·20?

Compound Proportion

Suppose a railway journey for 3 boys costs £1·65. For 5 boys it will cost £1·65 $\times \frac{5}{3}$ or £2·75. The cost varies in proportion to the number of fares. If the original journey was 60 miles, and assuming the cost is proportional to the distance travelled, we could find the cost of a rail journey of, say, 66 miles. The cost now depends not only on the number travelling, but also the length of the journey.

The original cost was £1·65

The number of boys increases in the ratio 5 : 3 or $\frac{5}{3}$

$\therefore$ The cost increases in the ratio $\frac{5}{3}$

The mileage increases in the ratio 66 : 60 or $\frac{11}{10}$

$\therefore$ The cost is again increased in the ratio $\frac{11}{10}$

$$\therefore \text{ The new cost is } 165 \times \tfrac{5}{3} \times \tfrac{11}{10}\text{p} = \overset{55}{\cancel{165}} \times \frac{\cancel{5}}{\underset{1}{\cancel{3}}} \times \frac{11}{\underset{2}{\cancel{10}}}$$

$$= \frac{605}{2}\text{p} = £3{\cdot}02\tfrac{1}{2}$$

This is a problem where two variables affect the original quantity, and is an example of *compound* proportion. In this case both the number of boys and the increase in the mileage were in direct proportion to the cost, but this is not always so.

Example:

Five bricklayers can build a wall 20 ft long in 10 hours. How long will 9 bricklayers take to build a wall 45 ft long and the same height, if they work at the same rate?

The original time was 10 hours.

Nine bricklayers build a 20-ft wall in less time than 5, and the new time *decreases* in the ratio 5 : 9 or $\frac{5}{9}$. (Inverse proportion: more men, less time.)

A wall 45 ft long will take more time than a 20-ft wall, and the time *increases* in the ratio 45 : 20 or $\frac{45}{20}$. (Direct proportion, more wall, more time.)

(i) If 5 men build a 20-ft wall in 10 h

$$\text{(ii)} \ \therefore \text{ 9 men build a 45-ft wall in } 10 \times \tfrac{5}{9} \times \tfrac{45}{20}\text{ h} = \frac{\cancel{10}}{} \times \frac{5}{\underset{1}{\cancel{9}}} \times \frac{\overset{5}{\cancel{45}}}{\underset{2}{\cancel{20}}}$$

$$= \tfrac{25}{2}\text{ h}$$

$$= 12\tfrac{1}{2}\text{ h}$$

Problems such as these can be written quite shortly. Decide what new quantity is being asked for (in this case, time), and write the first line as in (i), above, with the time at the right-hand side. Then, having decided whether the changes are direct or inverse, apply the ratios to the original quantity, as in (ii).

EXERCISE 7L

1. If 7 men earn £28 in 3 days, *how much* will 3 men earn in 9 days at the same rate?

2. If 7 men earn £28 in 3 days, *how long* will it take 8 men to earn £64, earning at the same rate?

3. If 7 men earn £28 in 3 days, *how many men* will be needed to earn £60 in 5 days, earning at the same rate?

4. An army barracks of 85 men uses 1250 kg of bread in 21 days. Forty-three men are sent on leave for 17 days. How much bread will be needed for those remaining during that time, if the average daily consumption per man is the same?

5. A hotel manager calculates that the cost of light, room cleaning and service was £45 for 18 rooms which had been occupied for 5 days by people attending a conference. How much should he allow for these costs if 105 rooms are occupied for 3 days and the daily average is the same?

6. A farmhand mowing a field of hay cuts 12 acres in 7 hours when the tractor's average speed is 4 mile/h. The next day he has to mow a field of 8 acres, but because of rough ground only averages 3 mile/h. How long will it take him?

7. To cover the floor of a room 16 ft long by 10 ft 6 in wide with carpet costs £54. What would it cost to cover a kitchen of the same width and 12 ft long with lino which is only one-third of the price of carpet?

8. Seven machines in a workshop operating for 8 hours a day take 19 weeks to turn a certain number of metal parts. How long would 4 machines working for $9\frac{1}{2}$ hours a day take to produce the same number of parts?

9. The weight of a bar of metal 90 cm long and 12 cm wide is 18 kg. Find the weight of a bar of the same metal 2 m long, 20 cm wide, and half as thick again.

10. If X men can earn Y pounds in Z hours, in how many hours will A men earn B pounds, at the same rate?

11. The Clerk of Works on a building site estimates that a certain job will take a gang of 8 men 6 days to complete. After 4 days 5 of the gang fall ill. How long will it take the remainder to finish the job?

CHAPTER 8

PERCENTAGE

THE WORDS "per cent" are Latin ones meaning "for every hundred," and in Roman times many taxes and transactions were calculated on so many units per hundred. Merchants since the Middle Ages have based their profits or losses on so much a hundred. Percentages are ratios in which the second term is 100, and may be expressed as fractions with denominators of 100.

If we say that 2 children in every 5 have fair hair, we are saying that $\frac{2}{5}$ of them have fair hair. This fraction $\frac{2}{5}$ can be written in the form $\frac{40}{100}$. We are then saying that 40 children in every 100 have fair hair—40 per hundred, or 40 per cent. The symbol % is used for "per cent" and so we write 40%.

In the chapter on ratios we saw that we could compare ratios by using the $n : 1$ or $\frac{n}{1}$ form, that is, by making their denominators equal to 1. Percentages are a way of comparing ratios by using a denominator of 100.

$$50\% = \tfrac{50}{100} \text{ or } \tfrac{1}{2}$$
$$30\% = \tfrac{30}{100} \text{ or } \tfrac{3}{10}$$
$$75\% = \tfrac{75}{100} \text{ or } \tfrac{3}{4}\text{, and so on.}$$

When, therefore, we talk about 75% of £4, we mean $\frac{3}{4}$ of it. Any fraction can be expressed as a percentage by changing it to an equivalent fraction with a denominator of 100:

$$\tfrac{1}{5} = \tfrac{20}{100} = 20\%$$
$$\tfrac{1}{3} = \frac{33\frac{1}{3}}{100} = 33\tfrac{1}{3}\%$$

and you should learn the following:

$50\% = \frac{1}{2}$	$33\frac{1}{3}\% = \frac{1}{3}$	$20\% = \frac{1}{5}$
$25\% = \frac{1}{4}$	$66\frac{2}{3}\% = \frac{2}{3}$	$10\% = \frac{1}{10}$
$12\frac{1}{2}\% = \frac{1}{8}$		$5\% = \frac{1}{20}$

Example:

Express $\frac{2}{15}$ as a percentage.

We must change $\frac{2}{15}$ to an equivalent fraction with denominator 100: Multiply numerator and denominator by 100,

$$\tfrac{2}{15} = \tfrac{200}{1500}$$

Divide both by 15, $= \dfrac{200 \div 15}{1500 \div 15}$

$= \dfrac{13\frac{1}{3}}{100}$

$= 13\frac{1}{3}\%$

Example:

Express $37\frac{1}{2}\%$ as a fraction.

$$37\tfrac{1}{2}\% = \frac{37\frac{1}{2}}{100}$$

$$= \tfrac{75}{200} = \tfrac{3}{8}$$

Example:

If 28% of a town's inhabitants are children, what percentage are adults?

For every 100 inhabitants, 28 are children.

$\therefore$ (100 — 28) are adults.

$\therefore$ 72% are adults.

Since decimals are also fractions, it is possible to express a decimal fraction as a percentage:

Example:

Express 0·634 as a percentage.

$$0{\cdot}634 = \frac{634}{1000}$$

$$= \frac{63{\cdot}4}{100} = 63{\cdot}4\%$$

EXERCISE 8A

Express these fractions with denominator 100, hence as percentages:

1. $\frac{1}{25}$	**3.** $\frac{2}{5}$	**5.** $\frac{5}{8}$	**7.** $\frac{7}{20}$	**9.** $\frac{1}{6}$
2. $\frac{3}{20}$	**4.** $\frac{3}{8}$	**6.** $\frac{7}{8}$	**8.** $\frac{1}{9}$	**10.** $\frac{5}{3}$

11. 92% of the boys at a school like ice-cream. What percentage does not? If the school has 300 pupils, how many would not like ice-cream?

12. Express as a percentage:

(a) 2 days in every 5 are sunny.

(b) 1 person in every 200 is involved in road accidents.

(c) 4 people in every 25 have motor-cars.

(d) 7 countries in every 8 are republics.

13. If I spend 95% of my income, what percentage do I save?

14. If 8% of the entrants in a beauty competition receives prizes, what percentage does not?

15. If 22% of the passengers on a train travels first-class, and another 6% first-class with sleepers, what percentage travels second-class?

16. Express the following percentages as fractions:

(a) 30% (c) 90% (e) 15%
(b) 70% (d) 85% (f) $62\frac{1}{2}$%

17. Express these percentages as fractions:

(a) 200% (c) 125%
(b) 320% (d) 245%

18. Express these decimal fractions as percentages:

(a) 0·15 (c) 0·04 (e) 0·591
(b) 0·37 (d) 2·47 (f) 0·703

If we wish to express one quantity as a percentage of another, they must first be in the same units:

Example:

Express 23p as a percentage of £3.

(1) Express 23p as a *fraction* of 300p:

$$\frac{23\text{p}}{300\text{p}}$$

(2) The percentage required is $\frac{23}{300} \times \frac{100}{1}\%$

$$= \tfrac{23}{3}\%$$
$$= 7\tfrac{2}{3}\%$$

Example:

Express 22 oz as a percentage of 1 lb.

(1) As a fraction: $\frac{22}{16}$

(2) As a %: $\frac{22 \times 100}{16} = \frac{\overset{11}{\cancel{22}} \times \overset{25}{\cancel{100}}}{\underset{\underset{2}{\cancel{4}}}{\cancel{16}}} = \frac{275}{2} = 137\frac{1}{2}\%$

Example:

Find $2\frac{1}{4}$% of £4·60.

$$2\tfrac{1}{4}\% = \frac{2\frac{1}{4}}{100}$$

$$\therefore 2\tfrac{1}{4}\% \text{ of } £4·60 = £4·60 \times \frac{2\frac{1}{4}}{100}$$
$$= £0·046 \times 2\tfrac{1}{4}$$
$$= 10·35\text{p} \simeq 10\tfrac{1}{2}\text{ p}$$

$2 \times £0·046 = £0·092$
$\frac{1}{4} \times £0·046 = £0·0115$
$\therefore 2\frac{1}{4}\% = £0·1035$

EXERCISE 8B

In No. 1-10 express the first quantity as a percentage of the second:

1. 3; 12.
2. 4 cm; 12 cm.
3. 1 ft; 1 yd.
4. 5 ft; 1 yd.
5. 7 cwt; 1 ton.
6. 3 in; 1 yd.
7. $4\frac{1}{2}$ in; 2 yd.
8. 2·3 g; 8 g.
9. 2 gal 5 pt; 3 gal 1 pt.
10. 320 ml; 2 litres.

In No. 11-16 find to the nearest halfpenny:

11. 5% of £12·25.
12. 8% of £33.
13. $2\frac{3}{4}$% of £47·50.
14. $5\frac{1}{4}$% of £2·67.
15. 7% of 87p.
16. 3·6% of £480.

17. 15% of $n = 12$. What is n?

18. 12% of all flights out of an airport are flown by Dutch airlines. If there are altogether 300 flights daily, how many are Dutch?

19. If the duty on a camera imported into this country is $33\frac{1}{3}$%, how much would have to be paid on a camera worth £54 before duty was charged?

20. Of 7,200 valves made in a factory, 4% were rejected on inspection. How many passed the inspection?

21. The population of Lisbon is 1,000,000 people, that of Portugal 9,000,000. What percentage of Portuguese live in the capital?

22. A bill at a restaurant came to £3·38. If it is customary to tip the waiter 15% of the bill, how much should he be given?

23. In a girls' school 68% of the girls in the last year passed the leaving examination and 16 girls failed. How many passed?

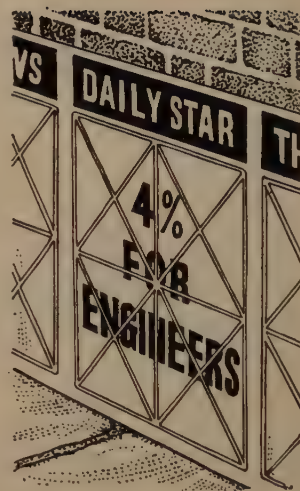
24. What percentage is x of y?

Percentage Changes

Every now and again we read of a trade union securing an increase in wages or salary for its members, and this is nearly always expressed as a percentage, "The engineers have just received a 4% increase in their wages," for example.

This would mean that for every 100 pence an engineer earned before, he is now going to earn 104, if he works for the same length of time. He is going to get an extra $\frac{4}{100}$ of his *original* wage, and the ratio, new wage

to the original, is 104 : 100, or as a fraction, $\frac{104}{100}$. You may remember that we called this fraction the multiplying factor. Knowing this, we could easily find his new wage, given the old.



Suppose his original wage was £10. Then his new wage would be £10 $\times \frac{104}{100}$, £10·40. The change in wages from £10 to £10·40 is expressed as a percentage.

Here is another change in value expressed as a percentage:

> "A man bought a motor-car and sold it two years later at a loss of 25%."

This is a statement which implies that the original price he paid is 100% and his selling price is 25% less, i.e. 75%. So in this case the multiplying factor would be $\frac{75}{100}$ or $\frac{3}{4}$. If he bought it for £600, then he would sell it for £600 $\times \frac{3}{4}$ or £450.

Changes of this kind, either up or down, are calculated by finding out what percentage the increase or decrease is of the *original* value.

Example:

If the rates on a cottage are £65 per annum and increase by 15%, what will the new rates come to?

The ratio of the new rate to the old is 115 : 100

$$\therefore \text{The new rate will be: } \frac{£65 \times 115}{100} = \frac{\overset{13}{\cancel{65}} \times \overset{23}{\cancel{115}}}{\underset{\underset{4}{\cancel{20}}}{\cancel{100}}}$$

$$\begin{array}{r} 23 \\ \underline{13} \\ 69 \\ \underline{230} \\ 4\overline{)299} \\ \hline 74\frac{3}{4} \end{array}$$

$$= £74{\cdot}75$$

Example:

If a boy in an examination scores 42 out of a possible 60 marks what percentage of the possible maximum did he score?

This can be thought of as a proportion sum:

$$60 \text{ marks} = 100\%$$
$$\therefore 42 \text{ marks} = 100 \times \tfrac{42}{60}\% = \frac{10\cancel{0}}{} \times \frac{\overset{7}{\cancel{42}}}{\cancel{6}\cancel{0}}$$
$$= 70\%$$

Example:

The number of men on a farm increased by 12% during a year. How many were there at first if there were 28 at the end of the year?

If the *original* number of men is thought of as 100% then the number at the end of the year must be equivalent to 112%.

$\therefore$ The ratio of the old number of men to the new is 100 : 112

$$\therefore \text{The original number of men} = 28 \times \tfrac{100}{112} \text{ men} = \frac{\cancel{28}}{} \times \frac{\overset{25}{\cancel{100}}}{\underset{\underset{1}{\cancel{4}}}{\cancel{112}}}$$

$$= 25 \text{ men}$$

We have already learnt that all measurement is inaccurate to some extent, and the idea of percentage error is a very useful guide to what degree of accuracy any particular measurement is made. The carpenter who constructs a door which is 1 in too tall, or short, has made a striking misjudgement—the door would not fit its frame, for a start.

On the other hand, the groundsman who makes an error of 2 centimetres when marking out a running track of 400 metres would not have made any noticeable error at all. These differences are reflected in the percentage errors. 1 in in 78 in (an average height for a door) is an error rather more than 1%. On the other hand 2 cm in 400 m is a relative error of $\frac{1}{20000} = 0{\cdot}005\%$ which is very much smaller.

Example:

A piece of canvas size 14 in × 10 in is wrongly assumed to have an area of 1 ft². Find the error per cent.

The true area is 140 in².

The false estimated area is 144 in².

$\therefore$ The error is 4 in².

$\therefore$ The error, *measured against the true area*, is

4 : 140, i.e. 1 : 35 or $\frac{1}{35}$

$\therefore$ The error per cent $= \frac{1}{35} \times 100\% = 2\frac{6}{7}\%$.

EXERCISE 8C

1. Increase 20 by 55%.

2. Decrease 225 by 5%.

3. Increase 208 by $37\frac{1}{2}\%$.

4. Decrease 120 by $62\frac{1}{2}\%$.

In No. 5-7 find n:

5. n is increased by 5% and then equals 42.

6. 17% of $n = 51$.

7. 30% of $n = 24$.

8. What is £55 increased by 80%?

9. What is £85 decreased by 20%?

10. What is £50 increased by 19%?

11. What is 2 ton 15 cwt. decreased by 40%?

12. If 1,404 pictures were sold at one exhibition of the Royal Academy, and this was an increase of 17% over the number sold in the previous year's exhibition, how many were sold in the previous year?

13. In a sale the cost of a shirt was reduced by 5%. What was the sale price if the original price was £2·25?

14. 48% of the workers in a car factory came out on strike. Altogether 1,440 workers struck. How many did not strike?

15. A man pays £135 per annum to a building society for the mortgage on his house. If this is 15% of his income, what does he earn?

16. The tax on a fur coat is $66\frac{2}{3}$%. If a woman pays £120·75 for such a coat, now much of this is tax?

17. 25% of a number is 78. What is $62\frac{1}{2}$% of the number?

18. At a by-election one candidate received 56% of the votes and had a majority of 3,960 over the other candidate. How many people voted?

19. A steel girder whose real length was 8 m was incorrectly measured as 7·6 m. What was the error per cent?

20. The distance between two towns is 156 miles, but the mileage recorded by a car odometer (distance recorder) is 158·4 miles. What, to one significant figure, is the error per cent?

Profit and Loss

Much of the trade of a country consists in buying goods of all kinds at one price and selling them at another. Your sweet shop will pay much less for its goods than it charges you, but in return the shop enables you to buy small quantities (which you could not do if you had to buy them from the manufacturer), and it is conveniently near.

To stock a shop of any kind costs a great deal of money, and it would scarcely be worth anybody's while to

sell goods worth £10 and receive only, say, 3p profit. On the other hand 3p would be a fair profit on something which cost only 10p.

So that in some way the profit has to be related to the cost of the articles sold. This is usually done by considering the profit (or loss) as a percentage of the cost price. If the Cost Price of an article is £100 and the Selling Price £101, the profit is £1.

If the Cost Price is £10, and the Selling Price £11, the profit is also £1.

But, in the first case, the profit is £1 as against a Cost Price of £100, i.e. 1% of the Cost Price. In the second case, the profit is £1 which is 10% of the Cost Price, a much better rate of profit. There is an important thing to notice. Profit per cent is *not* the profit on 100 articles but on the cost price, which can be considered as 100%.

In the examples and exercise which follows, C.P. = Cost Price, and S.P. = Selling Price.

Example:

If the C.P. of a chair = £12, and the S.P. = £13, what is the profit per cent?

The profit is £1.

This is $\frac{1}{12}$ of the C.P. $\left(\text{i.e. } \frac{\text{Profit}}{\text{C.P.}}\right)$

and $\frac{1}{12} = \frac{100}{12}\%$ or $8\frac{1}{3}\%$

Example:

If the C.P. of a portable radio is £18, and the radio shop makes a profit of 15%, what is the S.P.?

Consider the C.P. as 100%.

∴ The S.P. is 115%.

∴ The C.P. must be increased in the ratio 115 : 100 to arrive at the S.P.:

$$£18 \times \frac{115}{100} = £\frac{207}{10} \qquad \overset{9}{\cancel{18}} \times \frac{\overset{23}{\cancel{115}}}{\underset{10}{\underset{\cancel{20}}{\cancel{100}}}}$$

∴ The S.P. = £20·70

Example:

A car dealer sells a second-hand car for £245, and makes a profit of 40%. What did he pay for it?

If the C.P. is 100%, the S.P. in this case is 140%, and we must reduce the S.P. in the ratio 100 : 140 to arrive at the C.P.

$$\text{If £245 is 140\%, 100\% is £245} \times \frac{100}{140} = \frac{\overset{35}{\cancel{245}} \times \overset{5}{10\cancel{0}}}{\underset{7}{14\cancel{0}}} = 175$$

∴ C.P. = £175

Example:

A dress shop has a sale in order to clear the shop for new stock and a dress is sold for £8·75 at a loss of 12½%. What did it cost them?

If the C.P. is 100% and the loss is 12½%, the S.P. is 87½%, which in this case is £8·75.

∴ The S.P. must be increased in the ratio 100 : 87½ to arrive at the C.P.

$$\therefore \text{C.P.} = £8\tfrac{3}{4} \times \frac{100}{87\frac{1}{2}}$$

$$= \frac{35}{4} \times \frac{200}{175} = \frac{\overset{5}{\cancel{35}}}{\cancel{4}} \times \frac{\overset{\overset{2}{\cancel{50}}}{\cancel{200}}}{\underset{\cancel{7}}{\cancel{175}}}$$

$\therefore$ C.P. = £10

The important thing to remember in each case is that the C.P. is taken as 100% and *the profit or loss per cent is based on it.*

EXERCISE 8D

In No. 1-6, find the S.P.

1. C.P. £10, profit 5%.

2. C.P. £70, profit 20%.

3. C.P. £150, loss 33⅓%.

4. C.P. 2p, profit 25%.

5. C.P. 20p, loss 30%.

6. C.P. £280, profit 1%.

In No. 7-12, find the C.P.

7. S.P. £12, profit 20%.

8. S.P. £4, profit 60%.

9. S.P. £84, profit 5%.

10. S.P. £1·80, loss 25%.

11. S.P. £33, loss 12%.

12. S.P. 70p, profit 66⅔%.

In Nos. 13-22, find the profit or loss per cent.

13. C.P. £20, profit £5.

14. C.P. £550, profit £11.

15. C.P. £100, S.P. £94.

16. C.P. £8, S.P. £9.

17. C.P. 40p, S.P. 30p.

18. C.P. £1·75, profit 28p.

19. S.P. £36, profit £4.

20. S.P. £3·62½, loss 37½p.

21. S.P. £85, the C.P. £80.

22. S.P. 70p, C.P. 84p.

23. If I buy a painting for £240, and have to sell it at a loss of 60%, what was the S.P.?

24. A man buys a house for £2,200 and sells it some years later for £2,800. What is the gain per cent?

25. If apples are bought by the box, each holding 100, and a greengrocer buys them at £1·25 per box, what will his profit per cent be if he sells them for 2p each? (*Hint:* First work out the price per apple.)

26. A tractor is sold for £880 at a gain of 10%. Find the C.P.

27. A flat was bought for £1,500, and resold at a profit of 12%. What was the new S.P.?

28. A sports-shop sells a beach-ball at 63p, making a profit of 20%. What did it cost?

29. A man sold his motor-cycle for £62·50. This was a loss of $16\frac{2}{3}$%. How much did he originally pay for it?

30. A grocer buys butter by the cwt and pays £19·60 for it. How much per lb must he sell it to make a profit of 20%.

31. An ironmonger buys pegs at 30 for 10p and sells them at 2 for 1p. What is his profit per cent?

32. A shopkeeper would make 25% profit by selling a motor-mower for £35. What did it cost him, and how much must he sell it to make a profit of 20%?

33. A piano was sold for a profit of 35%, which was £87·50. What was the S.P.?

34. During a sale the price of a refrigerator was reduced from £63 to £52·50. What is the reduction per cent?

35. If an article costs £*n*, and is sold at a loss of 10%, what is its selling price in pence?

CHAPTER 9

SIMPLE INTEREST

NOWADAYS, it is quite usual for people to borrow, rather than buy, something they want to use. A housewife will hire a gas-cooker from the gas company, or a man will hire a car for a day, and naturally they have to pay for doing so. Many people pay to live in somebody else's house, and this payment is called rent. Sometimes money is hired in this way—a factory owner who wishes to build an extension to his works may borrow money from a bank to do so, or a man wishing to buy his house may borrow from a building society. The bank or building society will charge for the loan of the money.

On the other hand, if you have saved some money which you don't wish to use for the time being, a Savings Bank will pay you if you are prepared to lend it to them. The money remains yours, but they will pay you so much in addition for being allowed to use it. This payment for having the use of your money is called INTEREST, and depends on two things; firstly, how much you lend them and secondly, for how long.

The interest is usually expressed as a *percentage* of money lent. A building society might advertise that it is prepared to pay someone who lends them money 6% per annum. This means that they will give you £6 for the loan of every £100 (per cent—per hundred), for each year (per annum or p.a.), that it is lent. If you are prepared to lend £100 for 2 years they will pay you £12 (£6 for each year). If you are prepared to lend £200 for a year they will pay you £12 (£6 for each £100).

The same rules apply if money is borrowed. If our factory owner borrows £100 at 6% p.a. he has to *pay* £6 for each year that he borrows it (and, of course, pay back the money at the end of that time).

The money lent, or borrowed, is called the PRINCIPAL, and the percentage paid on it is called the RATE OF INTEREST.

If you lend a building society £100 and they pay you 6% p.a., at the end of a year you will own £100 (the principal) + £6 (the interest). The total sum, £106, is called the AMOUNT.

Example:

What is the interest on £300 for 3 years at 5% p.a.?

The interest on £100 for 1 year at 5% p.a. is £5.

∴ The interest on £300 for 1 year at 5% p.a. is £5 × 3.

∴ The interest on £300 for 3 years at 5% p.a. is £5 × 3 × 3 = £45.

Notice that this is a direct proportion sum.

If we lend money to a bank and they pay interest every year, so that the principal remains unchanged, we call this SIMPLE interest. Of course, it is possible to leave the interest in the bank with the principal so that during the next year interest is paid on the amount. This is called COMPOUND interest, because the interest earned is *compounded* with the original principal to make a new principal on which interest is calculated. But, in this chapter we shall deal only with Simple Interest, often abbreviated to S.I.

EXERCISE 9A

Find the Simple Interest (S.I.) on:

1. £100 for 1 year at 7% p.a.
2. £100 for 1 year at $2\frac{1}{2}$% p.a.
3. £100 for 4 years at 4% p.a.
4. £200 for 1 year at 3% p.a.
5. £300 for 5 years at 6% p.a.
6. £500 for 1 year at $3\frac{1}{2}$% p.a.
7. £250 for 1 year at 5% p.a.
8. £450 for 1 year at 6% p.a.
9. £300 for 5 years at 4% p.a.
10. £350 for 3 years at 2% p.a.

Finding the Amount

Example:

What is the amount at simple interest when £550 is borrowed for 4 years at $3\frac{1}{2}$% p.a.?

The interest on £100 for 1 year at $3\frac{1}{2}$% p.a. is £$3\frac{1}{2}$

∴ The interest on £550 for 1 year at $3\frac{1}{2}$% p.a. is £$3\frac{1}{2} \times \frac{550}{100}$

∴ The interest on £550 for 4 years at $3\frac{1}{2}$% p.a. is £$3\frac{1}{2} \times \frac{550}{100} \times 4$

$$= £\frac{7}{\cancel{2}} \times \frac{\overset{11}{\cancel{550}}}{\underset{2}{\cancel{100}}} \times \overset{2}{\cancel{4}} = £77$$

∴ The amount is £550 + £77 = £627

EXERCISE 9B

Find the amount at simple interest, of:

1. £100 for 12 years at 3% p.a.

2. £600 for 2 years at 8% p.a.

3. £350 for 4 years at $5\frac{1}{2}$% p.a.

4. £900 for $2\frac{1}{2}$ years at 5% p.a.

5. £250 for $4\frac{1}{2}$ years at $1\frac{1}{2}$% p.a.

6. £1300 for 1 year at 6%.

Finding the Principal

Sometimes the Amount, the Rate and the Time are known, and we wish to find the original Principal.

Example:

A certain sum of money is invested for 4 years at $3\frac{1}{2}$%. If it then amounts to £285, how much was originally invested?

The interest on £100 for 1 year at $3\frac{1}{2}$% is £$3\frac{1}{2}$
The interest on £100 for 4 years at $3\frac{1}{2}$% is $3\frac{1}{2} \times 4 =$ £14
£114 is the Amount for 4 years at $3\frac{1}{2}$% on £100, by proportion.
£285 is the Amount for 4 years at $3\frac{1}{2}$% on £$\dfrac{100}{1} \times \dfrac{285}{114}$
$=$ £250.

Notice that we begin by considering what the Amount would be on £100, and then we set out a proportion sum with the Principal on the right-hand side.

EXERCISE 9C

1. What sum would amount to £195 if invested for 10 years at 3%?

2. What sum amounts to £600 if invested for 16 years at $6\frac{1}{4}$%?

3. What is the Principal if the Amount after 6 years at $2\frac{1}{2}$% is £322?

4. What sum amounts to £30 if invested for 8 years at $2\frac{1}{4}$%?

5. Find the Principal if the Amount after 5 years at $4\frac{1}{2}$% is £33·07$\frac{1}{2}$.

The Formula for S.I.

Example:

What is the S.I. on £P for T years at R% p.a.?

As the interest on £100 for 1 year at 5% p.a. is £5, and at 8% is £8, so the interest on £100 for 1 year at R% p.a. is £R.

$\therefore$ The interest on £100 for T years at $R\%$ p.a. is £RT

and the interest on £1 for T years at $R\%$ p.a. is £$\frac{RT}{100}$

$\therefore$ the interest on £P for T years at $R\%$ p.a. is £$\frac{PRT}{100}$

Thus if $I =$ the Interest (in £),
$P =$ the Principal (in £),
$R =$ the Rate of Interest p.a.
and $T =$ the Time (in years),
then $I = \frac{PRT}{100}$ is the general formula for S.I.

When applying this formula it is important to remember that:

1. P is the number of pounds (or pounds and pence), in the principal.
2. T is the time in *years*. Any odd months or days must be expressed as a fraction of a year, which you should take as 365 days for this purpose.
3. R is the rate per cent per annum.
4. The resultant interest (I) is in pounds and pence.

Example:

What is the S.I. on £172 for 9 months at 5% p.a.?

If $I = \frac{PRT}{100}$,
$P = 172$,
$R = 5$
and $T = \frac{9}{12}$ or $\frac{3}{4}$ (*Note: T = time in years*).

Then substituting in the formula,

$$I = 172 \times \tfrac{5}{100} \times \tfrac{3}{4} = \frac{\overset{43}{\cancel{172}}}{} \times \frac{\overset{1}{\cancel{5}}}{\underset{20}{\cancel{100}}} \times \frac{3}{\underset{1}{\cancel{4}}}$$

$$= \frac{129}{20}$$

$$= 6{\cdot}45$$

$\therefore$ The S.I. = £6·45

Example:

Find the S.I. (to the nearest penny) on £84·33 from noon, March 12th to noon, June 5th of the same year at 6% p.a.

First find the time in days:

March 12th to March 31st	=	19 days
April	=	30 days
May	=	31 days
June 1st to June 5th	=	5 days
Total		85 days
	=	$\frac{85}{365}$ *years*

MARCH					APRIL				
31	3	10	17	24	..	7	14	21	28
..	4	11	18	25	1	8	15	22	29
..	5	12	19	26	2	9	16	23	30
..	6	13	20	27	3	10	17	24	..
..	7	14	21	28	4	11	18	25	..
1	8	15	22	29	5	12	19	26	..
2	9	16	23	30	6	13	20	27	..

MAY					JUNE				
..	5	12	19	26	30	2	9	16	23
..	6	13	20	27	..	3	10	17	24
..	7	14	21	28	..	4	11	18	25
1	8	15	22	29	..	5	12	19	26
2	9	16	23	30	..	6	13	20	27
3	10	17	24	31	..	7	14	21	28
4	11	18	25	..	1	8	15	22	29

$$\therefore T = \tfrac{85}{365}$$
$$P = 84{\cdot}33$$
$$R = 6$$

$$\therefore I = \frac{84{\cdot}33 \times 6 \times \overset{17}{85}}{\underset{73}{365} \times 100}$$

$$= \frac{8601{\cdot}66}{7300}$$

$$= \frac{86{\cdot}017}{73}$$

$$\therefore I \simeq 1{\cdot}178$$

$$\begin{array}{r} 84{\cdot}33 \\ 102 \\ \hline 168{\cdot}66 \\ 8433\phantom{{\cdot}66} \\ \hline 8601{\cdot}66 \\ \hline \end{array}$$

$$\begin{array}{r} 1{\cdot}178 \\ 73\overline{)86{\cdot}017} \\ 73\phantom{{\cdot}017} \\ \hline 130 \\ 73 \\ \hline 571 \\ 511 \\ \hline 600 \\ 584 \\ \hline \end{array}$$

$\therefore$ The interest is £1·18 (to the nearest penny)

EXERCISE 9D

Using the formula, find the S.I. on the following:

1. £250 for 3 years at 2% p.a.
2. £460 for 2 years at 3½% p.a.
3. £185 for 7 years at 7½% p.a.
4. £37·50 for 18 months at 4% p.a.
5. £120 for 2 years 6 months at 1¾% p.a.
6. £90·60 for 8 months at 25% p.a.

(*Questions* 7-13 *are to be worked as direct proportion sums, without using the formula.*)

Find to the nearest penny the S.I. on:

7. £52 for 3 years at $3\frac{1}{2}\%$.

8. £131·40 for $2\frac{1}{2}$ years at 2%.

9. £24·80 for $1\frac{1}{4}$ years at 4%.

10. £463·67$\frac{1}{2}$ for 4 years at $2\frac{1}{2}\%$

11. £178·39 for 100 days at 5%.

(*In questions* 12-16 *assume that the periods start and finish at noon on the dates specified.*)

12. £36 from April 1st to June 20th at $2\frac{1}{4}\%$.

13. £737·36 from September 13th to November 27th at $5\frac{1}{2}\%$.

Using the formula, find the S.I. on:

14. £270 from April 23rd to July 4th at $6\frac{1}{4}\%$.

15. £394·66 from October 27th to November 8th at 10%.

16. £146·59 from January 2nd to February 21st at $2\frac{1}{2}\%$.

17. What is the amount if £470 is invested for 5 years and Simple Interest is at the rate of 3%?

18. If I sell a house for £3,800 and invest the money with a building society at $6\frac{1}{4}\%$, what will they send me in interest each year?

19. A boy is given £4·50 on April 11th and he decides to open a National Savings Bank Account the same day with the money. How much will stand to his credit by December 12th if the Savings Bank pay $2\frac{1}{2}\%$ p.a. on each complete pound invested?

20. What is the S.I. on £a for b months at $c\%$ p.a.?

To Find the Rate Per Cent

$$\text{If } I = \frac{PRT}{100},$$

$$\text{then } 100I = PRT \text{ (multiplying both sides by 100),}$$

$$\text{and } R = \frac{100I}{PT} \text{ (dividing both sides by } PT\text{),}$$

and we can use this inversion of the formula to find the rate per cent if we know the principal, the time and the interest.

$$\text{Similarly } P = \frac{100I}{RT}$$

$$\text{and } T = \frac{100I}{PR}$$

What it amounts to is this. We have four variables, the Interest (I), the Principal (P), the Rate per cent p.a. (R) and the Time in years (T), and we can change the original formula around to express each one of these in terms of the other three. So that if we know any three we can find the fourth.

Example:

What rate of interest will earn £210 interest in 5 years on a principal of £840?

$$\text{If } R = \frac{100I}{PT},$$
$$I = £210,$$
$$P = 840$$
$$\text{and } T = 5,$$
$$\text{then } R = \frac{100 \times 210}{840 \times 5} = \frac{\overset{\overset{5}{\cancel{25}}}{\cancel{100}} \times \cancel{210}}{\underset{\cancel{4}}{\cancel{840}} \times \cancel{5}} = 5$$
$$\text{The rate} = 5\%\ \text{p.a}$$

Example:

What sum of money will yield £245 interest in 5 years at $3\frac{1}{2}\%$ p.a.?

Here we have to find the principal and so use the formula

$$P = \frac{100I}{RT}$$
$$\text{If } I = 245,$$
$$R = 3\tfrac{1}{2}$$
$$\text{and } T = 5,$$
$$\therefore P = \frac{100 \times 245}{5 \times 3\frac{1}{2}}$$
$$= \frac{100 \times 2 \times 245}{5 \times 7} = \frac{100 \times 2 \times \overset{\overset{7}{\cancel{49}}}{\cancel{245}}}{\cancel{5} \times \cancel{7}}$$
$$= 1400$$

$\therefore$ The principal required is £1,400.

Example:

How long will it take for £156 to earn £26 interest at 5% p.a.?

We have to find the time, so use the formula

$$T = \frac{100I}{PR}$$
$$\text{If } I = 26,$$
$$P = 169$$
$$\text{and } R = 5$$
$$\text{then } T = \frac{100 \times 26}{156 \times 5} = \frac{\overset{20}{\cancel{100}} \times \cancel{26}}{\underset{6}{\cancel{156}} \times \cancel{5}} = \frac{20}{6}$$

$\therefore$ The time required is $\frac{20}{6}$ years, or 3 years 4 months.

EXERCISE 9E

1. How long will it take for a principal of £360 to yield £48 in simple interest at 4% p.a.?

2. How much must be invested to produce £85 simple interest in 4 years at $6\frac{1}{4}$% p.a.?

3. If £125 is borrowed for 5 years and £25 interest is paid, what is the rate per cent per annum?

4. A factory owner borrowed £5,400 at $5\frac{1}{2}$% p.a. and paid £297 S.I. on his loan. For how long did he borrow it?

5. If I borrow £12 for 1 month and am charged 32p for the loan, at what rate per cent per annum is interest being charged?

6. A boy has some money in the National Savings Bank and receives interest at the rate of $2\frac{1}{2}$%. If the interest amounts to £1·33 after 8 months, how much money has he in the bank?

7. A bank manager tells a borrower that if he borrows £400 for 4 years he will have to pay £88 in S.I. At what rate per cent per annum is this interest being charged?

8. The interest on a loan at $3\frac{1}{2}$% for 2 years 6 months was £490. How much was borrowed?

9. If a bank lends *A* £200 at 4% p.a., *B* £300 at 6% p.a. and *C* £800 at 3% p.a., what is the total amount of interest it receives every year from *A*, *B* and *C*? What is the average rate of interest for the total amount it has lent?

10. If £36 amounts to £37 after 5 months, what will it amount to after 1 year?

CHAPTER 10

MENSURATION II

CONSIDER a square of side 1 ft. Its area is clearly 1 ft^2. But if we decide to work in inches, we must say that the sides of the square are each 12 in long, and so the area is

$$12 \times 12 \text{ in}^2$$
$$= 144 \text{ in}^2$$

Thus we have:

$$12 \text{ in} = 1 \text{ ft}$$
$$144 \text{ in}^2 = 1 \text{ ft}^2$$

Notice that

$$144 = 12^2$$

Now let us look at a cube whose edges are each 1 yd long, so that the volume is 1 yd^3. If we work in feet, the edges are 3 ft long, and so the volume is

$$3 \times 3 \times 3 \text{ ft}^3$$
$$= 27 \text{ ft}^3$$

Now we have

$$3 \text{ ft} = 1 \text{ yd}$$
$$27 \text{ ft}^3 = 1 \text{ yd}^3$$

Notice that

$$27 = 3^3$$

These examples indicate how to convert from one unit of area or volume to another.

First we take the normal table of length, such as

$$12 \text{ in} = 1 \text{ ft}$$
$$3 \text{ ft} = 1 \text{ yd, etc.}$$

If we are concerned with *area*, we now *square* these numbers:

$$(12^2)\ 144 \text{ in}^2 = 1 \text{ ft}^2$$
$$(3^2)\ 9 \text{ ft}^2 = 1 \text{ yd}^2$$

But if we are concerned with *volumes*, we *cube* the numbers in the original table:

$$(12^3)\ 1728 \text{ in}^3 = 1 \text{ ft}^3$$
$$(3^3)\ 27 \text{ ft}^3 = 1 \text{ yd}^3\text{, etc.}$$

EXERCISE 10A

Fill in the gaps in the following:

1. ( ) $mm^2 = 1\ m^2$.
2. ( ) $m^2 = 1\ km^2$.
3. ( ) $cm^3 = 1\ m^3$.
4. ( ) $in^3 = 1\ yd^3$.

Problems Concerning Areas and Volumes

Now suppose we wish to find the area of a rectangle 2 ft 3 in long and 1 ft 4 in wide.

Example

If we work in inches, we have

$$\begin{aligned} \text{area} &= 27 \times 16\ \text{in}^2 \\ &= 432\ \text{in}^2 \end{aligned}$$

If we want to convert this to square feet, we must divide by 144, thus:

$$\begin{aligned} \text{area} &= \frac{432}{144}\ \text{ft}^2 \\ &= 3\ \text{ft}^2 \end{aligned}$$

But we might have worked in feet from the beginning. The rectangle is $2\frac{1}{4}$ ft long and $1\frac{1}{3}$ ft wide, so

$$\begin{aligned} \text{area} &= 2\tfrac{1}{4} \times 1\tfrac{1}{3}\ \text{ft}^2 \\ &= \tfrac{9}{4} \times \tfrac{4}{3}\ \text{ft}^2 \\ &= 3\ \text{ft}^2\text{, as before.} \end{aligned}$$

This shows that the formula

$$\text{area} = \text{length} \times \text{breadth}$$

can still be used even when the lengths of the sides are expressed as fractions rather than whole numbers, so that squares of side 1 ft will not fit exactly into the rectangle.

Here is another problem, this time in metric units:

Example:

Find the volume of a box whose measurements are 2·4 cm by 2 cm by 1·5 cm (a) in cm^3; (b) in mm^3.

$$\begin{aligned} \text{(a) Volume} &= 2{\cdot}4 \times 2 \times 1{\cdot}5\ \text{cm}^3 \\ &= 2{\cdot}4 \times 3\ \text{cm}^3 \\ &= 7{\cdot}2\ \text{cm}^3 \end{aligned}$$

$$\begin{aligned} \text{(b) Volume} &= 24 \times 20 \times 15\ \text{mm}^3 \\ &= 24 \times 300\ \text{mm}^3 \\ &= 7200\ \text{mm}^3 \end{aligned}$$

Check:

$$\begin{aligned} \text{Since } 1\text{ cm} &= 10\text{ mm} \\ 1\text{ cm}^3 &= (10)^3 = 1000\text{ mm}^3 \\ \text{so } 7{\cdot}2\text{ cm}^3 &= 7{\cdot}2 \times 1000\text{ mm}^3 \\ &= 7200\text{ mm}^3 \end{aligned}$$

Units such as square feet and square yards are useful for measuring areas such as the floors of rooms, or football pitches, but tend to be inconveniently small when measuring areas of land.

A medium-sized field probably contains about 50 000 yd². To measure fields, plots of land, estates, etc., we use the *acre*.

A surveyors' chain is 22 yd long, so 1 square chain contains $22^2 = 484$ yd². An acre is defined as 10 square chains, and hence

$$1\text{ acre} = 4840\text{ yd}^2$$

For really big areas, like countries and deserts, the square mile is used.

$$\begin{aligned} \text{Since } 1\text{ mile} &= 1760\text{ yd} \\ 1\text{ mile}^2 &= 1760 \times 1760\text{ yd}^2 \\ &= \frac{1760 \times 1760}{4840}\text{ acres} \end{aligned}$$

$$1\text{ square mile} = 640\text{ acres}$$

(You should check the work by cancelling the fraction for yourself.)

EXERCISE 10B

1. Find in in³ the volume of a box measuring 2 ft 6 in by 10 in by 9 in. Convert this to ft³. Now calculate the volume again, working in feet, from the beginning, to see if you get the same answer.

2. The floor of a room is 5·2 m long and 3·3 m wide. The room is 2·6 m high. Find the area of the walls.

3. A football pitch is 110 yd long and 77 yd wide. Find its area in acres.

4. The cross-section of a beam is 10 in² and the beam is 24 ft long. Convert the area of the cross-section to ft², and hence find the volume of the beam in ft³.

5. A forest covers 3·6 square miles. How many acres is this?

6. The floor of a workshop is $22\frac{1}{2}$ ft wide and the total area is 810 ft². Find its length.

7. A rectangular flower-bed measures 3·5 m by 2 m, and there is a strip of grass 50 cm wide all round it. Find the area of the grass.

Problems of Greater Complexity

Here are three more examples of calculations involving areas and volumes. Study them carefully to see how to think out the solution, and also how to set out the working on paper.

Example:

Wallpaper is usually sold in strips (rolls) 12 yd long and 21 in wide. A man wants to re-paper his room, which he measures as follows:
Length 21 ft 6 in, Breadth 12 ft, Height from skirting board 7 ft 6 in. Area of doorway $19\frac{1}{2}$ ft², area of window 48 ft², area of fireplace $12\frac{1}{2}$ ft². The paper costs 65p per roll.

How much will it cost him altogether?

We need to know how many rolls he must buy. This can be found by finding the area to be papered, the area of a roll of paper, and doing a division sum. The area to be papered is found by subtracting the area of the doorway, etc., from the area of the four sides of the room. This latter can be found from the formula (perimeter) × (height): we will start here.

(Notice how we think backward from the answer required in order to find where to start.)

$$
\begin{aligned}
\text{Perimeter of room} &= 2 \times (21 \text{ ft } 6 \text{ in}) + 2 \times (12 \text{ ft}) \\
&= 43 + 24 \text{ ft} \\
&= 67 \text{ ft} \\
\text{Area of walls} &= 67 \times 7\tfrac{1}{2} \text{ ft}^2 \\
&= 502\tfrac{1}{2} \text{ ft}^2 \\
\text{Area unpapered} &= 19\tfrac{1}{2} + 48 + 12\tfrac{1}{2} \text{ ft}^2 \\
&= 80 \text{ ft}^2 \\
\text{Area to paper} &= 502\tfrac{1}{2} - 80 \text{ ft}^2 \\
&= 422\tfrac{1}{2} \text{ ft}^2 \\
\text{Area of roll} &= 36 \times 1\tfrac{3}{4} \text{ ft}^2 \\
&= 63 \text{ ft}^2 \\
\text{No. of rolls} &= \frac{422\frac{1}{2}}{63} \\
&= 7 \text{ (Since this comes to more than 6, he will have to buy 7 rolls.)} \\
\text{Cost} &= 7 \times 65\text{p} \\
&= \pounds 4{\cdot}55
\end{aligned}
$$

$$
\begin{array}{r}
67 \\
\underline{7\frac{1}{2}} \\
469 \\
\underline{33\frac{1}{2}} \\
502\frac{1}{2}
\end{array}
$$

$$
\overset{9}{\cancel{36}} \times \frac{7}{\cancel{4}} = 63
$$

$$
\begin{array}{r}
6{\cdot}7 \\
63\overline{)422{\cdot}5} \\
\underline{378} \\
445 \\
\underline{441} \\
4
\end{array}
$$

Example:

A concrete path 5 cm thick is to be laid to a width of 60 cm all round an ornamental pool 3m square. Find the volume of concrete required.

To find the volume, we must find the area of concrete and multiply it by the depth. To find the area, we use the subtraction method.

$$\begin{aligned}
\text{Outer area} &= 4{\cdot}2 \times 4{\cdot}2 \text{ m}^2 \\
&= 17{\cdot}64 \text{ m}^2 \\
\text{Inner area} &= 3 \times 3 \text{ m}^2 \\
&= 9 \text{ m}^2 \\
\text{Area of concrete} &= 17{\cdot}64 - 9 \text{ m}^2 \\
&= 8{\cdot}64 \text{ m}^2 \\
\text{Volume of concrete} &= 8{\cdot}64 \times \tfrac{5}{100} \text{ m}^3 \\
&= 0{\cdot}432 \text{ m}^3
\end{aligned}$$

Example:

The outside measurements of a shallow wooden drawer are 16 in 9 in and 2 in. The wood is everywhere $\frac{1}{2}$ in thick, and weighs 36 lb per ft³. Find the weight of the drawer.

Here we must find the volume of the drawer in *ft*³ and multiply by 36. It will be easier to start working in in³ and convert later.

To find the volume of the drawer, we will use an extension of the subtraction method which we used for areas.

We find the total (outer) volume of the box, including the wood, and then subtract from it the volume of the air inside the box (the inner volume). Since the wood is $\frac{1}{2}$ in thick, we find the length and breadth of the inside by subtracting 1 in ($\frac{1}{2}$ in at each end) from the outer measurements, but for the inner height we subtract only $\frac{1}{2}$ in, since a drawer has a base but no lid.

$$\begin{aligned}
\text{Outer volume} &= 16 \times 9 \times 2 \text{ in}^3 \\
&= 288 \text{ in}^3 \\
\text{Inner volume} &= 15 \times 8 \times 1\tfrac{1}{2} \text{ in}^3 \quad \text{in.} \\
&= 180 \text{ in}^3 \\
\text{Volume of wood} &= 288 - 180 \text{ in}^3 \\
&= 108 \text{ in}^3 \\
&= \tfrac{108}{1728} \text{ ft}^3 \\
\text{Weight} &= \tfrac{108}{1728} \times 36 \text{ lb} \\
&= 2\tfrac{1}{4} \text{ lb}
\end{aligned}$$

$$\frac{\overset{9}{\cancel{108}} \times \overset{\overset{1}{\cancel{3}}}{\cancel{36}}}{\underset{4}{\cancel{\underset{12}{\cancel{\underset{144}{\cancel{1728}}}}}}}$$

Notice how we did not cancel the fraction $\frac{108}{1728}$ ft³, but left it until we introduced the number 36, and then did all the cancelling at one go. This device can often save a lot of tedious multiplication and division.

EXERCISE 10C

1. A field 240 yd long and 110 yd wide was sold at a price of £187 per acre. How much did it cost?

2. A rectangular beam 10 m long, 20 cm wide and 8 cm thick is made of wood weighing 540 kg per m^3. Find its weight.

3. The inner measurements of a closed wooden box are 26 in, 16 in, 12 in. The wood is everywhere 1 in thick. Find the volume of wood used in making the box.

4. How many floorboards, each 8 ft long and 9 in wide, are needed for a room 20 ft by 12 ft?

5. A roll of wallpaper 12 yd long and 21 in wide costs 55p. How much will it cost to paper a room 21 ft long, 15 ft wide and 9 ft high, allowing 90 ft^2 for windows, etc?

6. A carpet 12 ft long and 9 ft wide is laid in a room 14 ft long and 10 ft wide, and the border is laid with lino at 20p per ft^2. How much will this cost?

7. A tank has a square base of side $2\frac{1}{2}$ m, and it contains $12\frac{1}{2}$ m^3 of water. What is the depth of the water?

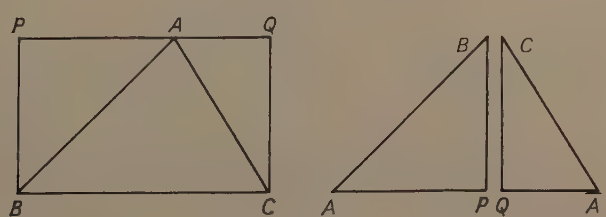
8. $1\frac{1}{2}$ in of rain falls on a roof of area 120 yd^2. How many cubic feet of water have fallen on the roof?

Geometrical Figures

Have a look at the following figures.

Example:

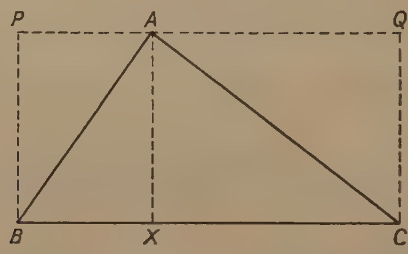
The accompanying diagram shows that if we cut off the two triangular pieces *PAB* and *QAC* from the rectangle *PQCB*, they will fit together to form a triangle which can lie exactly on top of triangle *ABC*.



In other words, the rectangle has been converted into two triangles, each of which has exactly half the area of the original rectangle. This enables us to devise a formula to find the area of a triangle.

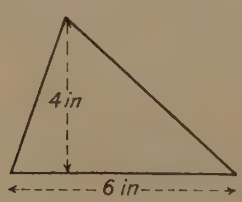
If we take a given triangle and turn it into a rectangle by adding the dotted line shown in the next diagram, we know that the area of the triangle is exactly half the area of the rectangle. Now the area of the rectangle is the length $\times$ the breadth, i.e. $BC \times BP$. The length BP is clearly equal to the length AX, so the area of the triangle is

$$\tfrac{1}{2} \times BC \times AX$$



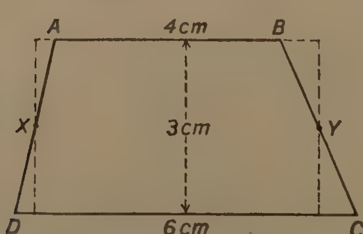
The line BC is called the *base* of the triangle, and the line AX is called the *perpendicular height*, or sometimes just the *height*. Thus we can put the formula in words:

(area of triangle) = (one half of base) $\times$ (perpendicular height)



For example, this triangle has an area

$$\tfrac{1}{2} \times 6 \times 4 \text{ in}^2$$
$$= 12 \text{ in}^2$$



The same sort of method can be used to find the area of a trapezium, which is a figure like the one shown in this diagram. The trapezium is the figure $ABCD$, in which the sides AB, DC are parallel, but of different lengths. The diagram shows that by cutting off the corners of the trapezium at the

mid-points of the sloping sides, i.e. at X and Y, and attaching them again higher up, we form a rectangle whose breadth is the perpendicular height of the trapezium, and whose length is XY.

Now the length XY is half-way between the lengths of the two parallel sides of the trapezium, and so is called the *average length.* Thus we have the formula

$$(\text{area of trapezium}) = (\text{average length}) \times (\text{height})$$

Example:

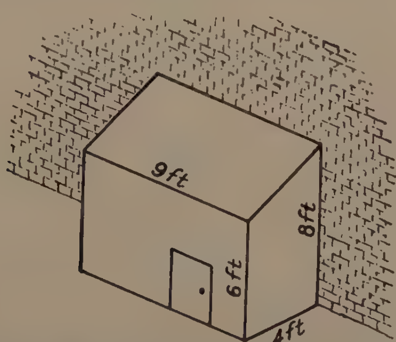
If we examine the dimensions of the given figure, we find that the two parallel sides are 4 cm and 6 cm long, so the average length is $\frac{1}{2}(4 + 6)$ cm $= 5$ cm. Since the height is shown as 3 cm, the area is

$$5 \times 3 = 15 \text{ cm}^2$$

In many problems we have to find the volume of a solid whose uniform cross-section is a triangle or a trapezium, and so we will have to use one of the two formulae above:

Example:

The diagram represents a garden shed built against the wall of a house. The front is 6 ft high, the back 8 ft high, and the shed is 4 ft deep and 9 ft long. Find the volume it encloses.



To find the volume we shall multiply the length (9 ft) by the cross-sectional area, that is, the area of the side of the shed, which is a trapezium. This trapezium is on its side, because the parallel sides are vertical. The "height" of the trapezium is, in fact, 4 ft.

$$\begin{aligned}
\text{Average length of cross-section} &= \frac{6+8}{2} \text{ ft} \\
&= 7 \text{ ft} \\
\text{Area of cross-section} &= 7 \times 4 \text{ ft}^2 \\
&= 28 \text{ ft}^2 \\
\text{Volume of shed} &= 28 \times 9 \text{ ft}^3 \\
&= 252 \text{ ft}^3
\end{aligned}$$

EXERCISE 10D

1. Find the area of a triangle whose base is 1 ft and whose height is 9 in.

2. A right-angled triangle has sides 5 in, 12 in, 13 in. Find its area. (Draw the triangle so that the 12 in side is the base: the longest side does not in fact come into the calculation.)

3. The parallel sides of a trapezium are 19 cm, 14 cm, and the height is 6 cm. Find its area.

4. An optical triangular prism is 18 cm long, and its cross-section is a triangle of base 2·5 cm and height 1·4 cm. Find its volume.

5. A trench 240 yd long is dug in a field. The cross-section of the trench is a trapezium 7 ft deep, 6 ft wide at the top and 5 ft wide at the bottom. Find the volume of earth removed in yd^3.

6. A swimming bath is 100 ft long and 30 ft wide. The depth increases uniformly from 4 ft at the shallow end to $9\frac{1}{2}$ ft at the deep end. Find the number of cubic feet of water in the bath.

Units of Capacity

We have seen that volumes are generally measured in ft^3, cm^3, etc. However, when we measure the volumes of liquids, such as water or petrol we often use the so-called units of capacity, such as quarts, gallons or litres. The metric system, as you have already seen, has a sensible connexion between the units of volume and capacity. 1 litre $= 1000\ cm^3$.

Thus it is very simple to move from one set of units to the other: a can 30 cm long, 24 cm wide and 40 cm deep has a volume of

$$30 \times 24 \times 40\ cm^3 = 2880\ cm^3$$
$$\text{Hence the capacity} = \tfrac{2880}{1000}\ \text{litres}$$
$$= 2{\cdot}88\ \text{litres}$$

There is no such simple conversion in English units, but a useful *approximate* relation is

$$6\tfrac{1}{4}\ \text{gal} = 1\ ft^3$$

Both systems are sensible when we come to converting capacity to weight. They both use water as a standard liquid, and we have the following facts:

1 gal of water weighs 10 lb
1 litre of water weighs 1 kg

EXERCISE 10E

1. A tank has a square base of side 4 ft and contains water to a depth of 2 ft 6 in. How many gallons is this? How much does the water weigh?

2. The cross-section of a pipe is 35 cm^2 and the pipe is 12 m long. Find how many litres of water it contains if full. What is the weight of this water?

3. Find the number of gallons in the swimming bath mentioned in Exercise 10D, No. 6, and find also the weight of this water in tons.

4. A petrol can, 15 cm × 12 cm in cross-section, contains petrol to a depth of 18 cm. If petrol only weighs $\frac{7}{10}$ as much as water, find the weight of the petrol in kg.

Formula for Depth of a Liquid

Since the volume of a solid of uniform cross-section is given by the formula

area of cross-section × length

it follows that to find the length we must divide the volume by the cross-sectional area. This device is very useful in finding the depth of a liquid in a vessel of uniform cross-section: it is given very simply by the formula

$$\text{depth} = \frac{\text{volume}}{\text{area of base}}$$

Example:

A tank $13\frac{1}{2}$ ft long and 10 ft 8 in wide contains 3825 gal. Find the depth of water in the tank.

We are given the capacity in gallons: this must be converted to a volume in ft³, and then we shall use the formula above, that is, we shall divide the volume in ft³ by the area of the base in ft².

To save a lot of heavy calculation, we shall leave all the actual working to the last possible moment, hoping that several numbers will cancel and simplify the work.

$$\begin{aligned}
\text{Volume of water} &= 3825 \div 6\tfrac{1}{4} \text{ ft}^3\\
&= 3825 \times \tfrac{4}{25} \text{ ft}^3\\
\text{Area of base} &= 13\tfrac{1}{2} \times 10\tfrac{2}{3} \text{ ft}^2\\
&= \tfrac{27}{2} \times \tfrac{32}{3} \text{ ft}^2\\
\text{Depth} &= (3825 \times \tfrac{4}{25}) \div (\tfrac{27}{2} \times \tfrac{32}{3}) \text{ ft}\\
&= 3825 \times \tfrac{4}{25} \times \tfrac{2}{27} \times \tfrac{3}{32} \text{ ft}\\
&= \tfrac{17}{4} \text{ ft}\\
&= 4 \text{ ft } 3 \text{ in}
\end{aligned}$$

Example.

One inch of rain falls on a roof of area 120 yd². This water is collected in a butt whose base has an area of 18 ft². Find the depth to which the butt will be filled.

We shall work in feet, and use the formula above once more.

$$\begin{aligned}
\text{Depth of rain} &= \tfrac{1}{12} \text{ ft}\\
\text{Area of roof} &= 120 \times 9 \text{ ft}^2\\
\text{Volume of rain} &= 120 \times 9 \times \tfrac{1}{12} \text{ ft}^3\\
&= 90 \text{ ft}^3\\
\text{Depth of water} &= \tfrac{90}{18} \text{ ft} = 5 \text{ ft}
\end{aligned}$$

EXERCISE 10F

1. A swimming bath is 50 yd long and 40 ft wide. 750 ft^3 of water are pumped in. How much does the water level rise?

2. A can 2 ft 6 in long and 1 ft 4 in wide contains 50 gal of water: find the depth of the water.

3. 480 kg of water are drawn out of a tank 1·2 m long and 0·5 m wide. How much does the water level fall?

4. One inch of rain falls over an area of 144 yd^2 and the water is collected into a rectangular tank which is 8 ft long and 6 ft wide. Find the rise of water in the tank.

5. A pipe delivers water to a square tank of side 4 ft at a rate of 10 gal per minutes. What will be the rise in the water level after one hour?

6. A tank with vertical sides contains water which has a surface area of 14 yd^2 and is everywhere 4 ft deep. Find to the nearest ton the weight of water in the tank. If the water is pumped out at the rate of 50 gal per minute, find the time required to empty the tank.

REVISION SUMMARY

ARITHMETIC

FRACTIONS I

$\frac{2}{3}$ is a proper fraction. *Page*

$\frac{7}{3}$ is an improper fraction.

$2\frac{1}{3}$ is a mixed number.

Cancelling: If we divide both top and bottom of a fraction by the same number, we get a simpler version of the same fraction.

Thus $\frac{15}{24} = \frac{5}{8}$ (cancelling 3). 12

Prime Factors: A number which divides into a second number without remainder is a factor of the second number. A number which has no factors (other than itself and 1) is a prime number. If a number is expressed as the product of several factors each of which are prime numbers, these are said to be its prime factors.

Indices: 2^3 means $2 \times 2 \times 2$ (8). This is read as "two to the power 3" or as "two cubed." The ^{3} is called an index or exponent. They can help to shorten the writing of prime factors. Thus to express 60 in prime factors we may write:

$$\begin{array}{r} 2\underline{)60} \\ 2\underline{)30} \\ 3\underline{)15} \\ 5 \end{array} \qquad \text{thus } 60 = 2^2 \times 3 \times 5$$

Express 1760 in prime factors. 14

Cancelling using prime factors.

$$\frac{132}{360} = \frac{2^2 \times 3 \times 11}{2^3 \times 3^2 \times 5} = \frac{11}{2 \times 3 \times 5} = \frac{11}{30}$$

Use prime factors to cancel $\frac{552}{828}$ 15

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Highest Common Factor: The largest number which is a factor of each one of a set of given numbers is called their H.C.F.
The H.C.F. of 132 ($2^2 \times 3 \times 11$) and 360 ($2^3 \times 3^2 \times 5$) is $2^2 \times 3$ (12) since two 2's and one 3 are the only factors common to both numbers.

Find the H.C.F. of 364 and 504. 16

Multiplication and Division: To multiply two fractions, multiply together the two numerators and the two denominators. To divide one fraction by another, turn the second upside down and then multiply. Mixed numbers must first be written as improper fractions.

Example:

$$3\frac{2}{3} \div 2\frac{1}{6} = \frac{11}{3} \div \frac{13}{6}$$

$$= \frac{11}{\cancel{3}} \times \frac{\overset{2}{\cancel{6}}}{13}$$

$$= \frac{22}{13}$$

$$= 1\frac{9}{13}$$

Divide $\frac{3}{4}$ by $\frac{7}{9}$. 19

MENSURATION I

Area: The area of a rectangle is given by the product of the length and the breadth. Both measurements must be in the same units.

Find the area of a rectangle 15 ft long and 12 ft wide:
(*a*) in ft²; (*b*) in yd²; (*c*) in in². 23

Inverse problems: $\text{Length} = \dfrac{\text{Area}}{\text{Breadth}}$

$$\text{Breadth} = \frac{\text{Area}}{\text{Length}}$$

Find the length of a piece of carpeting 3 ft. wide whose area is 180 ft². 24

Compound areas: These can be found by the addition or subtraction of simpler areas.

A photograph 9 in long and 7 in wide is mounted on a card so as to leave a border 1 in wide all round. Find the area of the border. 27

Perimeter: The distance round the edge of a plane figure is called its perimeter.

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Sides of a box: The area of the four sides of a room or box (without the base or lid) is given in total by the formula:

Area = Perimeter of base × Height

Find the area of the four sides of a box 6 in. × 4 in. × 3 in.

Volume: The volume of a rectangular box is given by the formula:

Volume = Length × Breadth × Height

All measurements must be in the same units.

Find in in^3 the volume of a box 2 ft × 1 ft × 5 in. 30

Cross-section: If a solid is cut at right-angles to its length, the resulting flat surface is called a cross-section.

If we get the same cross-section wherever we cut, we call the cross-section uniform.

The volume of a solid of uniform cross-section is given by the formula:

Volume = Area of cross-section × Length

PROPORTION, SPEED AND RATES

When quantities are connected so that when one increases the other does also (40 miles in 1 hour, 80 miles in 2 hours), they are in direct proportion.

When one decreases while the other increases (at 40 mile/h a journey takes 1 hour; at 20 mile/h the same journey takes 2 hours), they are in inverse proportion.

Some quantities are not connected proportionally at all.

Set out proportion sums so that the quantity you have to find is at the right-hand side.

If a car will travel 132 miles on 4 gal of petrol, how far will it travel on $7\frac{1}{3}$ gal?
(1st line: "*On* 4 *gal it travels* 132 *miles if it uses petrol at the same rate.*") 34

Six men are engaged to dig foundations of a house and are expected to take 10 days. Only 5 turn up for the job. How long should they take, if we assume that they work at the same pace? 35

If a centre-forward in a game of soccer scores one goal in 10 minutes, how many will he score in 90 minutes? 36

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The word "rate" is used to show the connexion between two quantities —money earned in 1 hour, cost for 1 ticket, etc. Speed is expressed as a rate (miles per hour, feet per second).

If destroyer A covers 600 nautical miles in 20 hours and destroyer B covers 558 nautical miles in 18 hours, which is the faster? 38

Average speed **is the total time divided by the total distance.**

What is the speed in feet per second of a cyclist who travels 4 miles in 12 minutes? 39

$$\textbf{Time} = \frac{\textbf{Distance}}{\textbf{Speed}},$$
$$\textbf{and Distance} = \textbf{Speed} \times \textbf{Time}$$

An aircraft covered the distance from Gander to Shannon (2240 miles) at an average speed of 280 mile/h. How long did it take? 40

A racing car completed a Grand Prix course in $2\frac{1}{2}$ h at an average speed of 112 mile/h. How many miles did it cover? 40

DECIMALS

Our system of using 10 numerals (including 0) and then combining them for larger numbers is only a convention. Number scales can be used with more or fewer numerals.

Can you say what 4, 7, 16 are on the *binary* scale? 42

Decimals are fractions whose denominators are powers of 10.

Write as decimals $1\frac{4}{10}$, $3\frac{6}{100}$, $2\frac{35}{100}$, 43

$\frac{1}{4}$, $\frac{3}{8}$, $\frac{7}{8}$ 46

Write as mixed numbers 40·06, 1·011 44

When adding or subtracting decimals, keep the decimal points underneath each other.

Find the value of: $27{\cdot}34 - 17{\cdot}85$ 46

$17{\cdot}28 + 6{\cdot}94 + 0{\cdot}06$ 46

$14{\cdot}27 - 3{\cdot}46 + 12{\cdot}701 - 5{\cdot}85$ 47

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It is impossible to measure absolutely accurately. A length measured as 4·2 or $4\frac{2}{10}$ in. to the nearest $\frac{1}{10}$ of an inch is "correct to 1 place of decimals." Measured as 4·23 in. to the nearest $\frac{1}{100}$ it is correct to 2 places of decimals.

What is 5·47286 correct to 4, 3, 2 and 1 dec. place? 49

To multiply decimals by 10, move the decimal point one place to the right; by 100, two places to the right and so on.
To divide, move the decimal point to the left.

Work out:

$2{\cdot}84 \times 10;\quad 3{\cdot}45 \times 100;\quad 8{\cdot}6 \div 10.$ 50

When multiplying decimals together, multiply the numbers together, ignoring the decimal points. Add together the total number of decimal places in the two numbers, count off this number from the right of the answer and put in the decimal point there.

Work out:

$14{\cdot}62 \times 3{\cdot}4;\quad 10{\cdot}104 \times 0{\cdot}0032;\quad 0{\cdot}025 \times 0{\cdot}04$ 53

When dividing one decimal by another, make the divisor a whole number by multiplying it by a power of ten. Multiply the dividend by the same number and set out the sum in long division form. The decimal point of the quotient must come immediately above that of the dividend. Make a rough check.

Work out $76{\cdot}59 \div 3{\cdot}7$
$2{\cdot}3085 \div 0{\cdot}045$ 54

Any fraction can be changed to a decimal by dividing the numerator by the denominator.

What are the following fractions as decimals?
$\frac{2}{5}$, $\frac{3}{8}$, $\frac{2}{7}$ (to 2 places). 55

FRACTIONS II

To add or subtract fractions we must express all of them with the same denominator.

Thus $\frac{3}{8} + \frac{5}{12} = \frac{9}{24} + \frac{10}{24} = \frac{19}{24}$.

Simplify: (*a*) $\frac{1}{4} + \frac{1}{3}$; (*b*) $\frac{7}{8} - \frac{1}{6}$. 57

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Mixed fractions: **Deal with the whole numbers first. It may be necessary to "borrow one" from the whole numbers when subtracting.**

$$\text{Thus:}\quad 5\frac{3}{8} - 2\frac{5}{12} = 3\frac{9-10}{24}$$
$$= 2\frac{33-10}{24} = 2\frac{23}{24}$$

Simplify: (*a*) $4\frac{3}{8} + 7\frac{11}{12}$;
(*b*) $4\frac{3}{8} - 2\frac{1}{12}$. 57

Lowest Common Multiple: **The smallest number into which several given numbers will divide without remainder is called the L.C.M. of these numbers. The L.C.M. of two or more numbers can be found from their prime factors, thus:**

$$56 = 2^3 \times 7$$
$$\text{and } 12 = 2^2 \times 3$$

and their L.C.M. requires all factors that occur in either number, i.e.

$$2^3 \times 3 \times 7\ (= 168).$$

Find the L.C.M. of 360 and 528. 58

Square roots: **$4 \times 4 = 16$: we say that 4 is the square root of 16. To find the square root of a large number, we express it in prime factors and then halve the index of each different factor.**

$$\text{Thus } 784 = 2^4 \times 7^2,$$
$$\text{so } \sqrt{784} = 2^2 \times 7$$
$$= 28.$$

Find $\sqrt{7056}$. 59

Several fractions: **When simplifying an expression containing more than two fractions, a given order must be followed:**

(*a*) deal with expressions in brackets,

(*b*) perform multiplication or division,

(*c*) lastly, do any other additions or subtractions.

Simplify: $\frac{2}{3} - \frac{1}{4} \times (\frac{1}{6} + \frac{1}{2})$. 60

"Double-decker fractions": **Simplify numerator and denominator of the main fraction separately, then divide the numerator by the denominator.**

Simplify: $\dfrac{\frac{1}{2} - \frac{1}{3}}{\frac{5}{8}}$ 61

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THE METRIC SYSTEM

The metric system is based on the metre, which is just over 1 yd long. Other units of length are found by multiplying or dividing it by powers of 10. Standard abbreviations are used: mm, cm, m and km.

A hectare (ha) is 10 000 m². Areas and volumes are expressed in the same way as British units, i.e. *square* metres, *cubic* metres.

Metric weights are based on the gramme which is the weight of 1 cm³ of pure water. Larger and smaller weights use the same prefixes as those used for length.

Compound quantities are not needed in the metric system. The litre **(1000 cm³) is the unit of capacity.**

Significant figures are those which must be kept, wherever the decimal point is, and are counted from left to right, beginning with the first digit which is not a zero. They are important in deciding what degree of accuracy has been used.

$$0{\cdot}003\ 25\ \text{km} = 3{\cdot}25\ \text{m} = 3250\ \text{mm}$$

The digits 325 are the significant figures here.

Express 0·004 073 to 3 sig. fig. 67

Express 52 700 to 2 sig. fig. 67

A measurement 9·7 in long (correct to 2 sig. fig.) means that the true length is between 9·65 and 9·75 in.

Two lengths are given as 5·6 and 3·4 in. Between what limits does their sum lie? **68**

Decimalization of non-metric quantities. In changing Imperial-measure quantities (e.g. gallons, quarts, pints; miles, furlongs, yards, feet, inches; cwt., quarters, pounds, ounces) into decimal fractions, change the lowest unit into the next higher (e.g. ounces into pounds, dividing by 16), work in decimals; add the higher unit and change into the next higher (e.g. pounds into quarters, dividing by 28); add the higher unit and change into the next higher (e.g. quarters into cwt., dividing by 4), and so on, *always working in decimals*. For example:

To express 1 mile, 3 furlongs, 6 chains as a decimal of 1 mile, change the chains to furlongs by dividing by 10, add the furlongs, change the furlongs to miles by dividing by 8 and add the miles.

Express 2 quarters 5 lb 8 oz as a decimal of 1 cwt (corr. to 3 places). 69

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RATIOS

The ratio of two quantities is the comparison made between them by expressing one as a fraction of the other.

Ratios, like fractions, are usually:

(1) Made as simple as possible.

(2) Expressed in the same units and as whole numbers.

Express these ratios in their simplest form:

30p : £1·80 71

$3\frac{1}{4}$ h : 20 min 71

$\frac{5}{6} : \frac{3}{4}$. 72

3·25 : 1·25. 72

To divide a quantity in a given ratio: e.g. £32 in the ratio 5 : 3. There are 8 (i.e. 5 + 3) parts, therefore each part must be £4 (32 ÷ 8). The first quantity is 5 of those parts (£20), the second 3 parts (£12). In other words, add the numbers in the ratio together and divide the total into the given quantity. Then multiply the quotient by each number in the ratio to obtain the various portions required.

Divide 28 cm in the ratio 3 : 4. 73

Divide 240 acres in the ratio 3 : 4 : 5. 74

Ratios can be combined if they have a common term.

If the length : width ratio of a room is 4 : 3, and the width : height ratio is 5 : 4, what is the ratio between the length : height? 75

$x : y$ as a ratio $= \frac{x}{y}$ as a fraction. The ratio of the new quantity to the original is called the multiplying factor.

Original quantity × multiplying factor = new quantity.

Increase 56p in the ratio 8 : 7. 77

Decrease 2 ft 8 in in the ratio 3 : 4.

The time of a train journey is reduced in the ratio 9 : 11 and the new time is 1 h 15 min. What was the original time? 77

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By what ratio must 2 ft 3 in be increased to become 6 ft? 78

Mrs. Jones cuts her consumption of biscuits by a quarter, but at the same time the price goes up by a quarter. If she formerly spent 40p a week on biscuits, what does she now spend? 79

Ratios can be compared by changing them to the form

$$n : 1$$
$$\text{or } 1 : n.$$

To convert $x : y$ to the form $n : 1$, divide x by y and put the result in place of n.

To convert $\frac{x}{y}$ to the form $1 : n$ divide y by x and put the result in place of n.

McKenzie saves 10p a week from his pocket money of 45p, and McKay saves $17\frac{1}{2}$p from 90p. For which is the ratio of saving to pocket money the greater? 80

The scale of a map or plan, in the form of $1 : n$ *or* $\frac{1}{n}$ is called the Representative Fraction (R.F.).

$$\text{R.F.} = \frac{\text{distance on map}}{\text{distance on ground}}$$

What is the R.F. of a map on the scale 4 cm to 1 km? 81

The R.F. of a map is $\frac{1}{10\,000}$. What distance on the ground is represented by 35 cm on the map? 82

Proportion sums can be solved directly by using ratios. Decide what new quantity is asked for, and write the first line with the original quantity at the right.
Decide whether the changes are direct or inverse, and apply the ratios to the original quantity.

A book is printed with 34 lines to a page.
It has 260 pages.
How many pages would be required if a larger type is used which only permits 30 lines to a page? 84

Five bricklayers can build a wall 20 ft long in 10 h. How long will nine bricklayers take to build a wall 45 ft long and the same height, working at the same rate? 86

PERCENTAGE

Page

"Per cent" means "for every hundred" and any fraction can be changed to a percentage by expressing it as an equivalent fraction with a denominator of 100.

$$\tfrac{2}{5} = \tfrac{40}{100} = 40\%$$

Express $\frac{2}{15}$ as a percentage. 88

$37\frac{1}{2}\%$ as a fraction. 89

0·634 as a percentage. 89

To express one quantity as a percentage of another, each must first be in the same units.

23p as a percentage of £3 = 23p as a percentage of 300p.
As a fraction:

$$\frac{23}{300}, \text{ and as a percentage}$$

$$\frac{23 \times 100}{300}$$

$$= 7\tfrac{2}{3}\%$$

Express 22 oz as a percentage of 1 lb. 90

Find $2\frac{1}{4}\%$ of £4·60 correct to the nearest penny. 90

Quantity changes can be expressed as percentages and are calculated by finding what percentage the increase or decrease is of the *original* value.

If the rates of a house are £65 p.a. and increase by 15%, what will the new rates come to? 92

If a boy scores 42 out of a possible 60 marks, what percentage of the maximum did he score? 92

The number of men on a farm increased by 12% during a year. How many were there at first if there were 28 at the end of the year? 93

Errors in measurement can be expressed as percentages.

$$\text{Error} = \frac{\text{Error} \times 100}{\text{True quantity}}$$

A piece of canvas 14 in × 10 in is wrongly taken as 1 ft². Find the error per cent. 93

Profit and loss is calculated as a percentage of the cost price (C.P.) which is taken as 100%, and *not* of the selling price (S.P.).

SIMPLE INTEREST

Interest **is payment for the use of money.**

Principal **is the amount borrowed (or lent).**

The Rate of Interest **is an annual percentage of the principal.**

The Amount **= Principal + Interest.**

Simple Interest **is the amount paid annually when the interest is not compounded with the principal.**

$\mathbf{I = \frac{PRT}{100}}$ **is the general formula for calculating S.I., where P is the principal (in £), R is the rate per cent p.a. and T is the time (in years).**

Before using the formula, remember to convert odd months or days to a fraction of a year.

The formula can be inverted so that

$$\mathbf{R = \frac{100\,I}{PT}},$$

$$\mathbf{T = \frac{100\,I}{PR}},$$

$$\textbf{and } \mathbf{P = \frac{100\,I}{RT}}$$

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What rate of interest will earn £210 interest in 5 years on a principal of £840? 104

What sum of money will yield £245 interest in 5 years at $3\frac{1}{2}\%$ p.a.? 104

How long will it take for £156 to earn £26 interest at 5% p.a.? 104

MENSURATION II

Change of units: To convert an area from one unit to another, the correct factor is the *square* of the corresponding length factor. Similarly, for volumes, the factor is the *cube* of the corresponding length factor.

$$12 \text{ in} = 1 \text{ ft}$$
$$144\ (12^2) \text{ in}^2 = 1 \text{ ft}^2$$
$$1728\ (12^3) \text{ in}^3 = 1 \text{ ft}^3$$

Land areas are measured in acres.

$$1 \text{ acre} = 4840 \text{ yd}^2$$
$$1 \text{ mile}^2 = 640 \text{ acres}$$

Volume of wood used to make a box. This is best found by calculating

(*a*) the total volume,
(*b*) the volume inside,
(*c*) and then subtracting.

The outside measurements of a shallow wooden drawer are 16 in × 9 in × 2 in. The wood is everywhere $\frac{1}{4}$ in thick and weighs 36 lb/ft³. Find the weight of the drawer. 110

Area of triangle. This is given by the formula

$$\text{Area} = \tfrac{1}{2} \times \text{base} \times \text{height}.$$

Area of Trapezium. This is given by the formula

$$\text{Area} = \text{Average length} \times \text{Height}$$

Capacity. Liquid volumes are often measured in these units:

1 litre = 1000 cm³	$6\frac{1}{4}$ gal = 1 ft³ (approximately)
1 m³ = 1000 litres	1 gal of water weighs 10 lb
1 litre of water weighs 1 kg	

Inverse problems.

$$\text{Height} = \frac{\text{Volume}}{\text{Area of base}}$$

A tank $13\frac{1}{2}$ ft long and 10 ft 8 in wide contains 3825 gal. Find the depth of water in the tank. 115

ALGEBRA

CHAPTER 1

LETTERS IN PLACE OF NUMBERS

THE WORD "algebra" comes from the Arabic *al-jabr*, meaning "the uniting of broken parts." This word was used by a famous Arab mathematician, Al Khwarismi, who was interested in calendars and the way in which the stars seemed to move across the sky. He lived eleven hundred years ago, so you can see that algebra has been studied for a very long time.

Even before that, some famous Greeks and Hindus had the idea of using letters or symbols to stand for unknown numbers. You already know some symbols. It is much quicker to write = instead of "*is equal to*" and + instead of "*add*", and algebra is a kind of shorthand, a way of saying things in mathematics much more shortly than would otherwise be possible. It is a kind of arithmetic with letters, and the important thing to remember is that, as far as possible, *letters are used in the same way as numbers.*

Addition

For example, suppose a boy is twelve years old. How old will he be in five years' time?

The answer is $12 + 5$ years $= 17$ years.

How old will he be in N years' time?

In the same way, the answer will be $12 + N$ years.

But we cannot add N to 12 until we know what number the letter N stands for. So we have to leave the answer as

$$(12 + N) \text{ years}$$

The brackets show that you are to consider $12 + N$ as a single number.

If you are T years old now, how old will you be in 5 years' time?

The answer is $(T + 5)$ years old.

If you are T years old, then in N years' time you will be

$$(T + N) \text{ years old.}$$

and in a further P years you will be

$$(T + N + P) \text{ years old.}$$

If you have 54 train numbers, and you spot x more, how many have you then?

Subtraction

We can subtract in the same way.

Example:

If there are 400 children in a cinema and 250 of them are boys, how many are girls?

The answer is $400 - 250 = 150$ girls.

If there are B boys, how many girls are there?

The answer is $(400 - B)$ girls

and, as before, we cannot subtract B from 400 until we know what number B stands for. So we have to leave the answer as $(400 - B)$ girls.

If there are C children altogether, of which B are boys, then there are

$(C - B)$ girls.

If a stick is 5 ft long and x ft are sawn off, how much is left?

Try these (remember that the letters stand for *numbers*, and as you have to write 8 *yd*, 4 *people*, 36 *sweets*, so you must say what *kind* of thing your letters represent: x *feet*, A *oranges*, d *metres*).

EXERCISE 1A

1. A man is 35 years old. How old was he 6 years ago? How old was he L years ago?

2. There are already 28 children in a classroom, and x more children come in. How many are there now? What is the answer if $x = 5$?

3. A flagpole is 12 m high, and b m are sawn off. How high is the pole now? What is the answer when $b = 2$?

4. There are 50 children going on holiday together. Only C children can get in the first coach. If all the children go in two coaches, how many must go in the second?

5. What is the total weight of 3 bars of chocolate, if one weighs 2 oz, one weighs 4 oz, and the third weighs g oz.

6. A wooden box and its lid weigh 6 lb. If the lid weighs y lb, how much does the box weigh? If the box and its lid weigh x lb, and the lid y lb, what does the box weigh?

7. 16 is an even number. The next even number above it is 18 $(16 + 2)$. If N is an even number, what is the next even number above? What is the one above that?

8. An aeroplane flies 600 miles, lands for refuelling, then flies another m miles, lands passengers, then flies another 450 miles. How far has it flown altogether? (Notice that 600 and 450 can be added together.)

9. What is the profit on a packet of sweets bought for 4p and sold for p pence? (Try it with numbers first.)

10. A jazz band has a pianist, a drummer, a double-bass player, x trumpeters and y saxophone players. How many are there in the band?

Multiplication and Division

Letters can be used in multiplication and division sums, just as we have used them in addition and subtraction sums.

Supposing that tennis balls cost 20p each:

6 tennis balls will cost 6×20p

a tennis balls will cost $a \times 20$p ($= 20 \times a$p)

This is usually written $20a$, and the multiplication sign is left out. Thus:

$$3 \times x = 3x$$

$$\text{and } 8 \times t = 8t$$

$$\text{but as } 1 \times 6 = 6 \times 1 = 6$$

$$\text{so } 1 \times y = y$$

We don't write $1y$. In the same way,

$$a \times b = ab$$

$$f \times g = fg.$$

If each of these apples cost 2p, what is the total cost?

This could be written $(2 + 2 + 2 + 2)$p.

But it is much quicker to write (4×2)p.

In the same way, if the apples cost x pence each, the total cost could be written

$$(x + x + x + x)\text{p}$$

But it is quicker to write $(4 \times x)$ pence or $4x$p

And once again we cannot get any further until we know what number x stands for. If $x = 5$, the answer is

$$4 \times 5\text{p (not 45, of course)} = 20\text{p}.$$

If I walk at 3 mile/h, in 2 hours I can walk 3×2 miles. How many miles can I walk in h hours?

EXERCISE IB

1. A car travels at an average speed of 40 mile/h. How far will it travel in t hours?

2. A baker sells x loaves at each house. When he has been to 14 houses, how many loaves has he sold?

3. A boy has 15 stamps on each page of his album. If there are n pages, how many stamps has he got?

4. If a shelf in the library is filled by 34 books, each x inches thick, how long is the shelf?

5. If a hall is 15 m long and b m wide, what is the area of the hall? (Remember the answer is in *square* metres.)

6. A watch loses x seconds a day. How much will it lose in a week? How much will it lose in the month of July?

7. A theatre has s seats. If each seat costs r pence, how much money will the theatre take if all the seats are sold?

8. How much would 1 cwt of grain cost at d pence per lb?

9. How much would a lb of toffees cost at w pence per oz?

10. How many pence are there in one pound? In three pounds? In b pounds?

Combining Letters and Numbers

Example:

We have seen that $12 \times n = 12n$	If $n = 4$, $12n = 48$.
In the same way $12 \times 3 \times n$	
$= 36 \times n = 36n$	If $n = 2$, $36n = 72$.
and $2y \times 4 = 2 \times y \times 4$	If $y = 3$,
$= 8 \times y$	$2y \times 4 = 6 \times 4 = 24$
$= 8y$	and $8y = 24$.
$2y \times 4t = 2 \times y \times 4 \times t$	If $y = 3$ and $t = 2$,
$= 8 \times y \times t$	$2y \times 4t = 6 \times 8 = 48$
$= 8yt$	and $8ty = 8 \times 2 \times 3 = 48$.
or better $8ty$.	

(Letters are usually put in alphabetical order.)

Similarly, $k \times l \times m = klm$ and $2p \times 3q \times 4r = 24\,pqr$.

(Notice that when letters and numbers are multiplied together, we usually put the number in front: $8y$ not $y8$.)

EXERCISE 1C

Work these out, and give your answer in the shortest form.

1. $a \times 2$

2. $e \times 5$

3. $1 \times b$

4. $7 \times 3 \times x$

5. $9 \times 6 \times b$

6. $4 \times y \times 8$

7. $3 \times a \times b$

8. $2 \times 8 \times c \times d$

9. $m \times l \times n$

10. $a \times 0$ (What is *any* number multiplied by zero?)

The Division Sign

Just as, with numbers, we can write $9 \div 3$ or $\frac{9}{3}$

so, with letters, we can write $9 \div x$, or $\frac{9}{x}$

and the second way is more usual in algebra.

$\frac{7}{a}$ means 7 divided by a. $\frac{b}{8}$ means b divided by 8.

$\frac{c}{d}$ means c divided by d.

How many centimetres are there in 90 millimetres?

Each centimetre is 10 millimetres, so there are $\frac{90}{10}$ c\metres.

How many centimetres are there in h millimetres?

There must be $\frac{h}{10}$ centimetres.

EXERCISE 1D

1. There are b exercise books in a pile which is 15 inches high. How thick is each book?

2. How many £ are there in 200p? In 600p? In xp?

3. If n boxes of matches cost 30p, what does one box cost?

4. How wide is an oblong of

(i) area 6 cm^2 and length 3 cm?
(ii) area 8 ft^2 and length x ft?
(iii) area A m^2 and length 4 m?
(iv) area A in^2 and length L in?

5. A man earns A pounds in a week of 56 working hours. How much an hour does he earn?

6. If you had A apples and ate half of them, how many would you have left?

7. If I buy the same article every day in November, and my bill at the end is r pounds, what is the price of each article?

8. A school which has p pupils is divided into equal classes, each one of which has r children. How many classes are there?

9. The carpet for a house cost £V, and there were $2w$ yd needed. How much per yard did the carpet cost?

10. A diesel train travelled 120 miles at an average speed of S miles/h. How long did it take?

"Translation" into Algebra

We have already remarked that algebra is a kind of shorthand. Over the page there are some examples to show you how to translate a sentence into algebra.

Sentence	*Algebra*
Multiply x by 7	$7x$
Add c to 4	$4 + c$ (or $c + 4$)
Subtract d from 8	$8 - d$ (NOT $d - 8$: why?)
Divide 5 by n	$\frac{5}{n}$
y added to 6 equals 9	$y + 6 = 9$ (can you guess what y is?)
Multiply x by 4, add 3 to the result, and then divide your answer by 6.	$\frac{4x + 3}{6}$

EXERCISE 1E

Write the following using symbols. Be careful to use capital letters where the question uses capitals, and small letters when they are used. (In some of the questions you may be able to find out what number the letters stand for.)

1. x plus 4 equals eight.

2. Twice a is equal to three.

3. Add four to b.

4. Take four from v. The result is equal to twelve.

5. Add five to y plus y.

6. Divide m by three: the answer is four.

7. Subtract eight times x from three times y.

8. B multiplied by 4 gives the same result as 15 minus B.

9. ($\therefore$ stands for "therefore") R divided by 2 is equal to 6. Therefore R is equal to 12.

10. Add 7 to three times d and divide the result by 11. The result is 2.

11. Multiply 3 by a and the result by b.

12. a and b are equal.

13. One half of W equals one.

14. Divide x by 6 and subtract the result from $\frac{1}{2}$.

15. Add m to n and divide the result by s.

Like and Unlike Terms

If Ronald has three pounds and his brother, Philip, has two pounds, then they have five pounds between them. We can add up $3 + 2$ here because we are talking about the same quantities in each case—pounds. But if poor Philip had only four pence, then we cannot add up three and four, because we are now talking about different quantities—pounds and pence. The usual way to write the answer is £3·04.

We can say that £3 + £2 = £5; but £3 + 4p cannot be simplified. We cannot add up different sorts of things.

Things of the same sort, like £3 and £2, are called LIKE TERMS, and like terms can be added (or subtracted). Things of different sorts, like £3 and 4p, are called UNLIKE TERMS, and cannot be added or subtracted until we know what numbers the symbols stand for. For example:

$2x + 5x = 7x$

but $2x + 5y$ has to be left as it is,

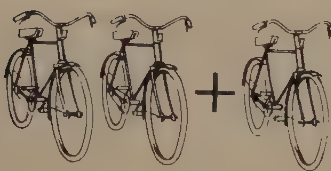
and so does $2x + 5$

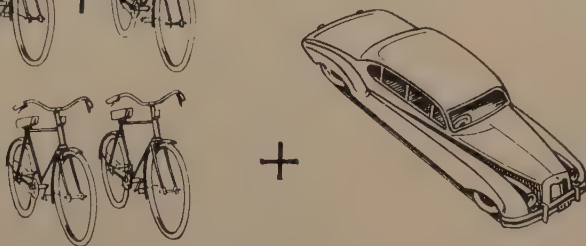
$8b - 3b = 5b$

$6a - a = 5a$ (remember a on its own means $1a$)

$7m + 4m - 3m = 8m$

$2c + 5c + 2d = 7c + 2d$ ($2c$ and $5c$ are like terms, and can be added, but $2d$ is unlike the others, and cannot).





2 bicycles + 1 bicycle = 3 bicycles.

2 bicycles + 1 car = 2 bicycles + 1 car.

EXERCISE 1F

Work out these examples in the same way. Make your answers as short as possible: if there is no shorter form, say so.

1. $m + m + m$
2. $v + v + v + v + v$
3. $a + a + a + a - a$
4. $b - b + b + b - b$
5. $f + f - f - f$
6. $3a + a$
7. $b + 5b$
8. $4m - m$
9. $7d + 2d$
10. $11x - 4x$
11. $3d + d + 2d$
12. $z + 3z + 5z$
13. $2x + 3x - 4x$
14. $5V + 5V$
15. $6M - 2M + 4M$
16. $a + 3a - 4a$
17. $3x + x + 5x - 3x$
18. $a + b$
19. $a + 2b$
20. $a + a + b$
21. $x + y + x$
22. $d + e + e + d$
23. $2d - 2c + 2d$
24. $2d + 2c - 2d$
25. $8h + 3g + 2h + 2g$
26. $6x - x + y$
27. $2m + 6 + 4m$
28. $3c + 4 - 2c + 7$
29. $7x - 2a - 2x$
30. $3q - 2r - r - 4r$

Evaluation

All of the examples in the preceding exercise are called EXPRESSIONS. Thus $a + 2b$ (No. 19) is an expression, and the separate parts, a and $2b$ are called TERMS of the expression (hence the phrases *like* and *unlike* terms). If we know what the unknown letters stand for, we can work out the value of the expression (this is known as EVALUATING it).

Example:

If $a = 3$ and $b = 2$
then in No. 19, $a + 2b = 3 + (2 \times 2)$
$= 3 + 4 = 7$

EXERCISE 1G

If $x = 4$ and $y = 3$, evaluate the following expressions:

1. $5x$
2. $x + 5$
3. $2x + 2$
4. $x + y$
5. $x - y$
6. $3x + 4y$
7. $3x - 4y$
8. xy
9. $3xy - y$
10. $5x - 6y - 1$
11. $\frac{x}{2}$
12. $\frac{1x}{2}$
13. $\frac{2x}{2}$
14. $\frac{y}{2x}$
15. $\frac{x}{4}$
16. $\frac{3x}{4y}$
17. $\frac{1}{x}$
18. $\frac{1}{xy}$
19. $\frac{y}{3xy}$
20. $2y - \frac{x}{y}$

Finding the Formula

You will know that to find the area of a floor you multiply the length by the breadth. We could write this:

The area is the length multiplied by the breadth.

Or, more shortly,

Area = length × breadth.

If we agreed to use A to stand for area,
l for length, and
b for breadth,
then we could write $A = l \times b$
or, as we have already learnt,

$$A = lb$$

This always applies. Whenever we want to find the area of an oblong, we multiply the length by the breadth. So $A = lb$ is a rule, or FORMULA. Such formulae (though usually much more complicated ones, of course) are continually used by scientists and engineers to help them to remember rules for calculating various important quantities in their work.

Suppose the floor is 18 ft long and 10 ft wide, then

$$l = 18 \text{ and } b = 10,$$
$$\text{so } A = 18 \times 10$$
$$= 180.$$

That is, the area is 180 ft^2.
If the measurements are in inches, and $l = 16$, $b = 8$, can you find the area?
If l and b are each 12, what shape is the floor?
If $l = x$ and $b = y$, then $A = xy$.
If we want to find how many inches there are in 8 feet, we multiply by 12.

In 8 feet there are 12×8 inches.
In 5 feet there are 12×5 inches.
In n feet there are $12 \times n$ or $12n$ inches.

Here is another formula: n ft $= 12n$ in.
(Which means that, whatever number n stands for, 12 times that number tells you how many inches are in that number of feet.)

In 1 ton there are 20 cwt.
In 3 tons there are 20×3 cwt.
In t tons there are $20t$ cwt,
that is, t tons $= 20t$ cwt.

Can you say how many minutes in y hours?
How many pence in x pounds?
(If you are not sure, *try it with numbers first*.)

Similarly

$$24 \text{ in} = \frac{24}{12} \text{ ft}$$
$$= 2 \text{ ft}.$$
$$60 \text{ in} = \frac{60}{12} \text{ ft}$$
$$= 5 \text{ ft}.$$
$$a \text{ in} = \frac{a}{12} \text{ ft}$$

(Say "a over 12" feet: we cannot work it out any further until we know what a stands for.)

Can you say how many tons in 80 cwt?
How many tons in y cwt?
How many hours in c minutes?
How many £ in y p?

EXERCISE 1H

Try these:

1. How many tons are there in:
(i) a cwt? (ii) s lb?

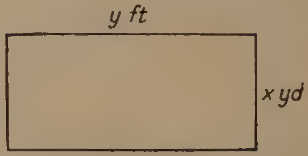
2. How far will a racing cyclist travelling at 25 mile/h go in 2 h? In 5 h? In z h?

3. How many pence are there in £1? In £4? In £x?

4. How many centimetres are there in 1 metre? In b metres? In x kilometres?

5. How many ounces are there in 6 lb? In z lb? In $\frac{z}{2}$ lb?

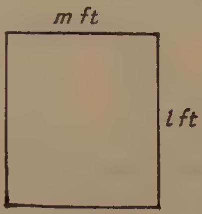
6. How many grammes in m kilogrammes?



7. What is the area of this floor? (As with numbers, both the lengths must be in the same units. How are you going to change x yd to feet?)

8. If n is an odd number, what is the next odd number below it?

9. How many weeks in $2a$ days? In y years?



10. What is the perimeter of this oblong field? (Like terms can be added together.)

More about Multiplication

We have already seen that $3 + 3 + 3 + 3 + 3 = 5 \times 3$, and $a + a + a + a + a = 5a$. This can be a great saving in time and trouble. Imagine $200a$ written out as $a + a + a + a + \ldots$!

There is a similar way of dealing with a large number of multiplications. $3 \times 3 \times 3 \times 3 \times 3$ can be written 3^5, which is pronounced "three to the power five." $a \times a \times a \times a$ is written a^4 (a to the power four). Don't mix up the two symbols:

$4a$ means four a's added together.
a^4 means four a's multiplied together.

Thus $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$.

Area and Volume

The area of a square of side 6 cm is (6×6) cm²

$$= 6^2 \text{ cm}^2$$
$$= 36 \text{ cm}^2$$

and the volume of a cube of edge 4 cm is $(4 \times 4 \times 4)$ cm³

$$= 4^3 \text{ cm}^3$$
$$= 64 \text{ cm}^3$$

In the same way, the area of a square of side x cm is x^2 cm^2
and the volume of a cube of edge y cm is y^3 cm^3.

Because of this connection with the area of a square and the volume of a cube, x^2 is usually pronounced "x—squared" and y^3 is pronounced "y—cubed," instead of the longer phrases "x to the power two" and "y to the power three."

An expression like $2a^2$ means $2 \times a^2$
$= 2 \times a \times a$.

It does NOT mean $2a \times 2a$.

Similarly, $3a^3b$ means $3 \times a^3 \times b$.

Since $3 \times 3 = 9$, we say that 9 is the SQUARE of 3
and 3 is the SQUARE ROOT of 9.

What is the square of 4? What is the square root of 4?

There is a special symbol for square root, thus: $\sqrt{\ }$.
$\sqrt{25}$ means the square root of 25. (What is it?)
$\sqrt{x}$ means the square root of x.

EXERCISE 1J

If $x = 7$ and $y = 9$, what are the values of:

1. x^2
2. y^2
3. $y^2 - x^2$
4. $\sqrt{y}$
5. $\sqrt{(x + y)}$
6. $\sqrt{x^2}$
7. $2x^2$
8. $3x^2y$
9. $(x + y)^2$

If also $z = 2$, evaluate these:

10. x^2yz
11. $2xy^2z$
12. $4xyz^3$

Simplifying Expressions

You have learnt that $2y \times 4 = 8y$
and $2y \times 4x = 8xy$.

Examples:

$$\begin{aligned} 3a \times 2a &= 3 \times a \times 2 \times a \\ &= 6 \times a \times a \\ &= 6a^2 \end{aligned}$$

$$\begin{aligned} \text{and } 4y \times 2x \times 3 &= 4 \times y \times 2 \times x \times 3 \\ &= 24xy \end{aligned}$$

$$\begin{aligned} \text{and } 2b \times b^2 &= 2 \times b \times b \times b \\ &= 2b^3 \end{aligned}$$

$$\begin{aligned} \text{and } 2b^2 \times d^3 &= 2 \times b^2 \times d^3 \\ &= 2b^2d^3 \end{aligned}$$

What is $3m^2 \times 2n$?

EXERCISE 1K

Simplify the following:

1. $s \times s^2$
2. $2t \times t^2$
3. $3ab \times 2$
4. $cd \times d$
5. $cd \times cd$
6. $2a^3 \times 2$
7. $ab \times ba$
8. $xy \times yz$
9. $2uv \times 4vw$
10. $2x^3 \times 3y^2$

The Cancellation of Fractions

We have seen that $\frac{a}{12}$ means $a \div 12$. Here are some further instances.

Example:

$$\frac{9m}{3} \text{ means } \frac{9 \times m}{3}, \text{ that is, } 9 \times m \div 3$$

so that $\frac{9 \times m}{3}$ can be cancelled thus: $\dfrac{\overset{3}{\cancel{9}} \times m}{\underset{1}{\cancel{3}}}$

$$= \frac{3m}{1} = 3m.$$

Example:

Also, $\dfrac{5ab}{15ab} = \dfrac{5 \times a \times b}{15 \times a \times b}$

and this can also be cancelled $\dfrac{\overset{1}{\cancel{5}} \times \overset{1}{\cancel{a}} \times \overset{1}{\cancel{b}}}{\underset{3}{\cancel{15}} \times \underset{1}{\cancel{a}} \times \underset{1}{\cancel{b}}} = \dfrac{1 \times 1 \times 1}{3 \times 1 \times 1} = \dfrac{1}{3}$

Here are some more examples:

(i) $\dfrac{15p^2}{5p} = \dfrac{\overset{3}{\cancel{15}} \times p \times \overset{1}{\cancel{p}}}{\underset{1}{\cancel{5}} \times \underset{1}{\cancel{p}}}$

$$= \frac{3 \times p \times 1}{1 \times 1}$$

$$= 3p$$

(ii) $\dfrac{m}{m} = \dfrac{\overset{1}{\cancel{m}}}{\underset{1}{\cancel{m}}} = 1$

(Remember that, as $\frac{3}{3} = 1$, so $\frac{m}{m} = 1$, NOT 0.)

Always be careful not to mix up subtraction and division.

$$6a - 2a = 4a.$$

$$\text{BUT } \frac{6a}{2a} = 3$$

EXERCISE 1L

Try these:

1. $8x \div 2$
2. $20a \div 10$
3. $8x \div 2x$
4. $9ab \div 3ab$
5. $8x \div x$
6. $\dfrac{a^2}{a}$
7. $\dfrac{4a^2}{2a^2}$
8. $\dfrac{3a^2b}{a^2}$
9. $\dfrac{7ab^3}{2b^3}$
10. $\dfrac{3x^3}{9x}$

The next exercise has both multiplication and division sums. Write your answers in the shortest form.

EXERCISE 1M

1. $a \times a^2$
2. $m \times 2n$
3. $hk \times k$
4. $de \times df$
5. $8b \times 1$ (What is any number multiplied by one?)
6. $3b \times 3b$
7. $4k \times 0$ (What is any number multiplied by zero?)
8. $\dfrac{2d}{d}$
9. $\dfrac{g}{g}$
10. $\dfrac{3x}{3}$
11. $\dfrac{3}{3x}$
12. $\dfrac{8m \times 2}{2}$
13. $\dfrac{7d}{1}$
14. $\dfrac{6f}{9}$
15. $\dfrac{9a}{3}$
16. $\dfrac{5b \times 2}{4}$
17. $\dfrac{gh}{hg}$
18. $\dfrac{a}{a^2}$
19. $\dfrac{gh}{g^2}$
20. $\dfrac{b}{3} \times 3b$
21. $\dfrac{2a \times b^3}{2}$
22. $\dfrac{4m}{m^2 \times 2}$

Algebra is a language, and in this first chapter we have been learning the meanings of some of the important basic words of this language. Without them we cannot hope to make progress, any more than a small child can hope to make progress in arithmetic until he has learnt his multiplication tables. But once we have mastered these basic words, the meanings of the various symbols of algebra, we can go on to learn how to use algebra as a tool which can do a thousand jobs—and mathematicians are still finding new uses for it.

CHAPTER 2

FORMULAE

We saw in Chapter 1 that Algebra was a kind of shorthand. We all use some abbreviations—perhaps there are too many nowadays, but it does save time. Everybody uses BBC for British Broadcasting Corporation, LP for Long-Playing and TUC for the Trades Union Congress. Algebra is full of such short cuts—they mean much less writing and leave more time for thinking. But why not use machines to do our calculating?

At first sight that sounds even quicker, and a great deal of arithmetic, adding, multiplying and so on, is done by simple calculating machines. Banks and business firms use more complicated machines called computers. But it would be impossible to find even a computer that could tell *straight away* for example, how thick the cable on a crane must be to lift three-ton loads without breaking. If it was too thin it would be very dangerous if the cable broke, someone might be killed, and if it was too thick it would be wasteful and the crane would have to carry far too big a weight of cable. Somebody has to experiment in order to find out, and his discoveries are fed into the computer.

Formulae for Building

Men have been making such experiments for thousands of years, with all sorts of materials. About the year 1100 A.D. some builders, for example, made churches with stone walls and roofs. At first they made the walls very thick by using lots of large stones. As time went on they thought that perhaps they could try thinner walls, and make higher roofs, and some of our large churches and cathedrals have almost no walls, but only piers and buttresses, with glass windows instead of walls. Of course, they were not very scientific or mathematical at first about their experiments and sometimes their buildings fell down when they became too bold and built too high, or with too little stone. Gradually, by watching the results of their experiments they evolved rules which they could use in other buildings later, with perfect safety. Nowadays, architects and engineers have many rules which allow them to build, for example, long concrete bridges only six inches thick in the centre.

Of course, it would be an awful waste of time to have to work out one's own rules for everything. The man who designs a crane does not have to test varying thicknesses of cable to see which is the most suitable. He finds the rule, which other engineers and mathematicians have worked out, and

applies it. Each law, or rule, is called a *formula*. This is a Latin word and the correct plural is *formulae*. The great advantage of formulae is that although the experiments to work them out are sometimes complicated, using them is usually easy. Have you ever dropped a stone down a well, and waited for the splash at the bottom? Someone has worked out the connexion between the time it takes for the stone to drop and the distance it falls. It looks like this:

The distance in feet = the time in seconds, squared, × 16.

This is rather a mouthful, and difficult to remember, so by letting

s = the distance in feet
t = time in seconds

the formula became $s = 16t^2$.

Example:

Suppose with a stop-watch you had timed the stone's fall, and it was 2 seconds, we could use this formula to find how deep the well is:

t (the time in seconds) is 2,

$$\therefore s = 16 \times 2^2$$
$$= 16 \times 4$$
$$= 64 \text{ ft.}$$

(How deep would the well be if the stone took 4 seconds to drop?)

The interesting thing is that this formula can be used for all falling objects. A parachutist (who was a mathematician!) could jump from his aircraft at 10 000 ft and wait 20 seconds before pulling the ripcord. He would have fallen 16×20^2 ft, or 6400 ft, so he would still have 3600 ft for his parachute to open, and modern airmen, flying at great heights where it is very cold, have to do just this and delay the opening, to avoid being frozen on the way down. (As a matter of fact, there are other things he would have to take into account as well, so don't try it till you know some more mathematics.) What the formula does for us is to tell us how to find what we want —it gives us instructions. This particular formula tells us:

> "You want to find how far the stone (or man) will fall? Then put down the time where it says t. Square it, multiply the result by 16, and the answer will be the distance in feet."

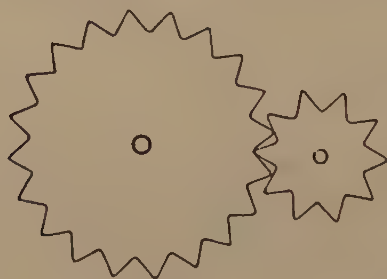
The trouble about using other people's formulae is that it is only too easy to apply them parrot fashion. If you want to learn to think in algebra, there is nothing quite as good as making your own formula. It is really a kind of pattern finding. We have all come across some patterns already in numbers:

If 1 foot = 12 inches
2 feet = 24 inches
3 feet = 36 inches
8 feet = 96 inches

The pattern is easy, each foot is the same as 12 inches. To make a formula, use a letter (any one, as long as we stick to it) to stand for the *number* of feet, and multiply it by 12 to find the number of inches.

$$n \text{ feet} = 12n \text{ inches.}$$

How you experiment will depend on what you are interested in. We could experiment using a pair of gear wheels. Supposing you have one with 20 teeth and another with 10 teeth. How many times will the small one turn for each whole revolution of the big one? If you mark one tooth on each you will



20 tooth cog *10 tooth cog*

soon discover that the small one turns twice for each complete turn of the large. Now we have three numbers, 10, 20 and 2 (wheels). How do 10 and 20 give us the answer 2? It looks like $20 \div 10$ or $\frac{20}{10}$. Try it with two more cogs, 12 for the smaller and 36 for the larger. This time the smaller revolves three times. Now we have 12, 36 and the answer 3 times. This looks like 36 divided by 12. Is there any pattern connecting the two results? In the first the larger (20) divided by the smaller (10) gave the right answer. In the second the larger (36) divided by the smaller (12) gave the right answer. So it looks as if to find the number of times the smaller revolves we must divide the number of teeth on the larger by that on the smaller. Let us use letters for shortness and call L the number of teeth on the larger, S the number of teeth on the smaller; and A the answer.

Then our formula will read $A = \frac{L}{S}$

And, of course, the useful thing is that having found a formula for this problem we can use it for *any* numbers of teeth on any two cog wheels. If you have a bicycle you can find easily how many times the rear wheel revolves for each turn of your chainwheel.

Let us try to make another formula. Suppose you are going by car from London to Sheffield (which is 160 miles) and the journey takes 8 hours (not including stops). The average speed is obviously 20 mile/h. The question

to ask yourself is *how* did I get that answer? You will see that you divided the distance (160 miles) by the time (8 hours). If you then travelled from Sheffield to Exeter (240 miles) and, the roads being clearer, only took 6 hours (once again, not including stops), your average speed would obviously be 40 mile/h. Asking yourself the same question—"*How* did I work it out?"—the answer is the same. The distance divided by the time gave the average speed. This looks like a general rule or formula:

$$\text{Average speed} = \frac{\text{distance}}{\text{time}}$$

Using our shorthand, so that v = average speed in mile/h
d = distance in miles
t = time in hours

we have a formula: $v = \dfrac{d}{t}$

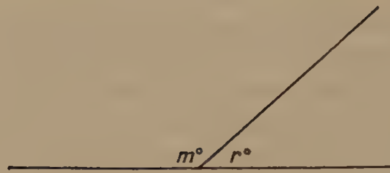
(You may be puzzled by one thing. The letter d for distance is obvious, and so is t for time, but why v for speed? The fact of the matter is that *any* letters will do for your own abbreviations, but they don't become useful to other people unless everybody understands what each letter in a formula stands for, and v in this well-known formula stands for "velocity." Both capitals and small letters are sometimes used in formulae. There is no difference in meaning. We could have written $V = \dfrac{D}{T}$; the important thing is that once we have chosen a letter, small or capital, we stick to it.)

So the way to make your own formula is:
(i) Find out *how* a special problem is solved.
(ii) Make a general rule.
(iii) Abbreviate it with letters.

Sometimes the letters are given to you. Suppose you are asked to write down the formula for the height (H ft) of a pile of 17 books, each of which is A inches thick.

If one book is A inches thick,
then 17 books are $17A$ inches thick.
But we are asked to find the height in *feet*.
To change inches to feet, divide by 12.
$\therefore$ The height *in feet* is $\dfrac{17A}{12}$
But we are told that H is the height in feet.
So the formula is $H = \dfrac{17A}{12}$
We call this a formula *for H in terms of A*.
In the same way $v = \dfrac{d}{t}$ is a formula *for v in terms of d* and t.

Example:



Find a formula *for m in terms of r.*

The angles on a straight line add up to 180°

$$\therefore\; m^\circ + r^\circ = 180^\circ$$

$$\therefore\qquad m = 180 - r$$

(If you are in doubt about the letters, try it with numbers.)

EXERCISE 2A.

Write the formula for:

1. The cost (£A) of N stamps at 3p each.

2. The weight (w oz) of 18 pencils each weighing x oz.

3. The quantity (v gal) of z bottles of milk each containing 1 pint.

4. The cost (£a) of staying 6 days in a hotel at £c per day.

5. The cost (£b) of staying n days in a hotel at c *pence* per day.

6. The distance (d miles) travelled by a space station orbiting the earth in 12 hours at b miles per hour.

7. The time taken (t hours) by a boy cycling 4 miles to school at v miles per hour.

8. The speed (x miles per hour) of a train travelling from London to Edinburgh (370 miles) in y hours.

9. The profit (p pence) on y plants bought by a nurseryman at x pence each and sold by him at $2x$ pence each. (How much does he make on each plant?)

10. The time (t seconds) taken by a boy running 100 yd at v ft per second.

The Use of Brackets

Before we go on to make more interesting and complicated formulae, we must learn something about the use of brackets. Let's see how they can save time. Have a look at the diagram on the opposite page.

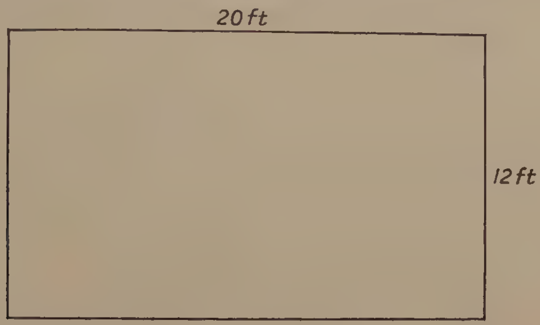
Suppose we want to find the perimeter of this oblong room. There are two sides of 20 ft and two of 12 ft.

$$\therefore \text{ the perimeter is } 20 \times 2, + 12 \times 2 \text{ ft}$$

$$= 40 + 24 \text{ ft}$$

$$= 64 \text{ ft}$$

In other words we have done *two* multiplications (20×2 and 12×2) and *one* addition ($40 + 24$) in order to arrive at the answer.



Now, if we use brackets, and wrap up the length and breadth in one parcel, as it were, we can write:

$$\begin{aligned}\text{The perimeter} &= 2 \times (20 + 12) \text{ ft}\\ &= 2 \times 32 \text{ ft}\\ &= 64 \text{ ft.}\end{aligned}$$

This time we have done only *one* addition ($20 + 12$), and *one* multiplication (2×32). You will find that lots of time can be saved by using brackets. This useful idea was thought of quite late in the history of mathematics, and was first used in a book published in 1629.

The important thing to remember is that the contents of a bracket are to be treated as a single number:

$(4 + 8)$ means the number obtained by adding 4 to 8;

$(x + 3)$ means the number obtained by adding 3 to x;

and $2 \times (20 + 12)$ means multiply by 2 the number obtained by adding 12 to 20.

It is customary to leave out the multiplication sign. Instead of $2 \times (20 + 12)$, we write $2\,(20 + 12)$.

$3\,(12 + 17)$ means, add 17 to 12, and multiply the result by 3;

$2\,(y + 5)$ means, add 5 to y and multiply the result by 2;

and $x\,(a + 4)$ means, add 4 to a, and multiply the result by x.

EXERCISE 2B

Write down expressions, using brackets, for these:

1. $x + y$, multiplied by 2.
2. Add m to $2n$, and multiply the result by 3.
3. Subtract 4 from a, and multiply the result by x.
4. Four times the sum of s and t.

5. The product of a and $n - 1$.

6. A man pays 25p to see a football match, and he has to take 2 bus rides at x pence each to get there and back. If he took his wife with him, what did the afternoon cost him?

Of course, the contents of a bracket can be divided, as well as multiplied.

$(9 - 1) \div 2$ means, take 1 from 9 and divide the result by 2.

We can also write this in two other ways:

(a) $\frac{1}{2}(9 - 1)$, $9 - 1$ multiplied by $\frac{1}{2}$ (which is the same as dividing by 2).

(b) $\dfrac{9-1}{2}$. In this form no brackets are necessary. The line serves the same purpose.

EXERCISE 2C

Write down expressions, using brackets, for these:

1. Divide $a + 3$ by 4 (3 answers). **2.** Half the sum of $n + m$.

3. One-fifth of the difference between a and b.

4. The number of kilogrammes in $(x + y)$ grammes. (How are grammes changed to kilogrammes?)

5. The number of tons in $a + c + 2d$ cwt.

6. The number of millilitres in $(2a + 40)$ litres.

7. The number of yards in $(8 - t)$ inches.

8. The product of x and the number 3 larger than x.

9. A plank is 12 ft long. A length N ft is cut off, and the remainder divided into 5 equal parts. How long is each part?

10. Subtract x from y and divide the result by 3.

The trouble with working out an expression like this: $x(a + 4)$, is that we cannot add a and 4 because they are unlike terms. Let us see whether we can get round this difficulty by going back to our first example of brackets: $2(20 + 12) = 64$. It is easy to see that we can get the same answer by multiplying each number in the bracket by 2, and adding the products:

$$\begin{aligned} 2 \times 20 &= 40 \\ 2 \times 12 &= \underline{24} \\ \text{TOTAL} &= 64 \end{aligned}$$

We can make a picture of it like this:

$$2 \times 20 = 40$$
$$2\,(20 + 12) \quad 40 + 24 = 64$$
$$2 \times 12 = 24$$

Of course, this scarcely seems worthwhile with numbers, but in this example, $2(x + 3)$, we cannot add the terms in the bracket, because they are unlike terms, and we have to use this method.

$$2 \times x = 2x$$
$$2(x + 3) \rightarrow 2x + 6$$
$$2 \times 3 = 6$$

Similarly, $a(4 + y) = 4a + ay$.

Example: Work out $3(2a + 5c)$

$$3(2a + 5c) = (3 \times 2a) + (3 \times 5c)$$
$$3 \times 2a = 6a$$
$$3 \times 5c = 15c$$
$$\text{TOTAL} = 6a + 15c$$

Multiply *each* term in the bracket by the multiplier.

EXERCISE 2D

Work out these examples:

1. $2(x + y)$.
2. $5(4a + b)$.
3. $x(3 + y)$.
4. $3(2d + e + 2f)$.
5. $a(b - c)$.
6. $2a(x + 2y)$.
7. $8d(6 + f)$.
8. $7(5y + z + 9)$.
9. $3(x^2 + x)$.
10. $a(a + a^2)$.

You should now be able to work out the values of these expressions: If $x = 2$, $y = 3$ and $z = 4$, find the value of the following examples. (In each of the examples 11-14, you ought to substitute numbers for letters first, and then remove the brackets. Check them by removing the bracket first and substituting afterwards):

11. $3(y - x)$.
12. $x(y + 2)$.
13. $5(x + y - z)$.
14. $3y(2x + 3z)$.
15. $\frac{4(x + y)}{2}$.
16. $(x + y)^2$. (*Hint:* Add x and y and square the result.)
17. $x^2 + y^2$. (Notice that this is quite different from No. 16.)
18. $\frac{1}{3}(x + 2)$.
19. $x + y(z - x)$. (This means, subtract x from z, multiply the result by y and add the answer to x.)
20. $y^2 + 3(x^2 - 1)$.

Construction of Formulae

Now that you understand how to use brackets, we can use them to make more interesting formulae. Remember that we have to find a pattern.

Example:

A popular cookery book suggests that a good rule for cooking meat is to keep it 15 minutes in the oven for each pound and add 20 minutes to the total time. If a housewife buys a joint weighing L pounds and cooks it for M minutes, find a formula for M in terms of L.

If the joint weighs 2 lb, she must cook it for $(2 \times 15) + 20 = 50$ minutes.
If the joint weighs 5 lb, she must cook it for $(5 \times 15) + 20 = 95$ minutes.
So that if the joint weighs L lb, she must cook it for

$$(L \times 15) + 20 \text{ or } 15L + 20 \text{ minutes.}$$

But we are told she cooks it for M minutes, so

$$M = 15L + 20.$$

Of course, it is important to apply common sense. If this rule was applied to a tiny piece of meat weighing one ounce, so that it had to be cooked for nearly 21 minutes, it would probably be overdone! Such rules only apply within a limited range.

Example:

A telegram cost 25p for the first 12 words and 2p each for each additional word. If the cost of a telegram containing n words (n being greater than 12) costs p pence, find a formula for p in terms of n.

The telegram costs 25p anyway, for 12 words.
So the number of words left over (since there are more than 12) is n (the total number of words) less 12, or $n - 12$.

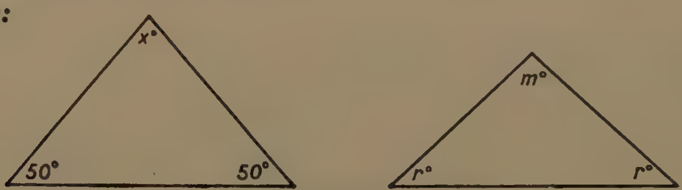
These cost 2p each $\therefore$ $n - 12$ words cost $2(n - 12)$ pence.
The first 12 cost 25p.
The rest cost $2(n - 12)$ p.
$\therefore$ The total costs $25 + 2(n - 12)$ pence.
We can work out this bracket.

$$25 + 2n - 24$$
$$= 1 + 2n, \text{ which is the total cost in pence.}$$

Since the telegram costs p pence,

$$p = 1 + 2n$$

Example:



In the second triangle, find a formula for m in terms of r.
The angles of a triangle add up to 180°.

In the *first* triangle $x = 180°$ minus the other two

$$= 180° - (50° + 50°)$$
$$= 180° - 100°$$
$$= 80°$$

Likewise, in the *second* triangle

$$m = 180° - (r° + r°)$$
$$= 180° - 2r$$

$\therefore$ the formula for m in terms of r is

$$m = 180 - 2r$$

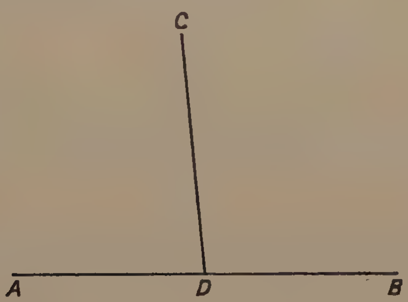
EXERCISE 2E

Try these for yourself:

1. What is the perimeter of a rectangular field if it is 40 yd long and 25 yd wide.

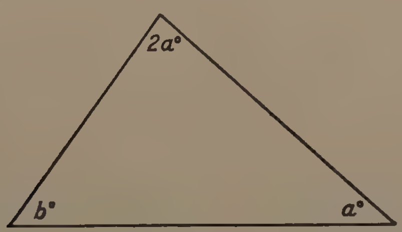
If the perimeter is p yd, the length is x yd, and the width y yd, complete the formula $p =$? (p in terms of x and y).

2.



If ADB is a straight line (and $A\hat{D}B$ therefore 180°), and $A\hat{D}C$ is 85°, what is $C\hat{D}B$? If $A\hat{D}C$ is $x°$ and $C\hat{D}B$ is $y°$, express x in terms of y.

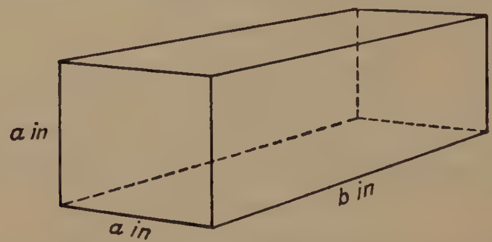
3.



If the sum of the angles of a triangle is 180°, find a formula for b in terms of a.

4. A bus has s seats, each taking 3 passengers, and t seats each taking 2 passengers. Write down a formula for A, the total number of passengers the bus can take.

5. If $a = 3$ and $b = 5$ find the volume of this box. If the volume of the box is C in^3, complete the formula $C = ?$ (C in terms of a and b).



6. A man is left £120, which he puts into the bank. If he saves £30 a year thereafter, and takes nothing out, how much will he have in 5 years time? If he has £x after y years, write a formula for x in terms of y.

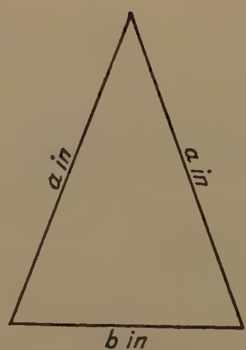
7. Mr. Jones is taking his car abroad, and decides to go by air. He finds that it costs £7 to take his car to Calais, and each passenger costs £3. What will it cost him to take his car, his wife, two friends and himself to Calais? If the airline company earns £A for each flight, carrying n cars and p people, write a formula for A in terms of n and p.

8. Oranges cost 3 for 5p. If x oranges cost m pence, find a formula for m in terms of x.

3 cost 5p $\qquad$ 1 costs $\frac{5\text{p}}{3}$ $\qquad$ x cost?

9. The perimeter of this triangle is 11 in. Find a formula for:

(i) a in terms of b;
(ii) b in terms of a.



10. A pond was drained for cleaning. 100 gallons of rainwater fell during the night, and in the morning when the hydrant was turned on to refill it, water flowed in at the rate of x gallons an hour. How much water was in the pond after 48 hours? After m hours? If there were G gallons in the pond after h hours, complete the formula $G = ?$ (G in terms of x and h).

11. Letters abroad cost 4p for the first ounce, and 2p for each additional ounce. What is the cost, in pence, for a letter weighing 4 oz? What is the cost, in pence, for a letter weighing T oz? (Work out the bracket.)

12. What is the cost of 4 oz of sweets at 3p per ounce? Of x oz at 3p per ounce?

A confectioner mixes b oz of sweets at 2p per ounce with c oz at 3p per ounce. What is the total cost of the mixture? What is the cost of 1 oz of the mixture? If the cost per ounce of the mixture is C pence, find the formula for C in terms of b and c.

13. In this series of numbers 1, 5, 9, 13, 17

the 2nd number 5 is $(1 + 4)$
the 3rd number 9 is $[1 + (4 \times 2)]$
the 4th number 13 is $[1 + (4 \times 3)]$
the 5th number 17 is $[1 + (4 \times 4)]$

What is the nth number?

14.

$$3000 \text{ m} = 3 \text{ km}$$

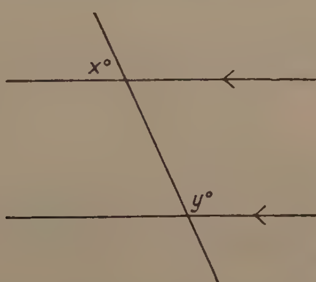
$$8000 \text{ m} = 8 \text{ km}$$

If x m $= y$ km, find a formula for y in terms of x.

15. (i) A bicycle travels 7 ft for each complete revolution of its wheels. How far does it travel in 9 revolutions?

(ii) The wheel of a car, travelling D m in one revolution, turns E times. If the car travels C m complete the formula $C =$? (C in terms of D and E).

16.

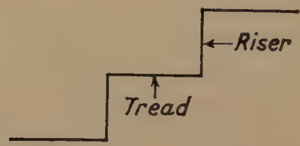


Find a formula for x in terms of y. (See page 301 for the properties of angles on parallel lines.)

17. A yacht travels at b knots in still water. How fast will it travel *with* a current of c knots? *Against* the same current? Remembering the formula $\text{Time} = \dfrac{\text{distance}}{\text{speed}}$, how long will the yacht take to sail d nautical miles *with* the current? And how long will the return journey *against* the current take? If T hours is the total time there and back, what is the formula for T in terms of b, c and d?

Substitution in a Formula

Most of our useful formulae; how to work out the horse-power of a car, how to convert miles to kilometres, and so on, have been worked out for us, and are ready for us to use. In this case we take the general rule and replace the letters by the numbers in our particular problem. There is a useful formula used by builders for working out the relative size of the risers and treads of staircases.



Most of us have walked up or down stairs where this rule has not been applied, and noticed how we seem to strike the next tread with our foot before, or after, we expect it. It can be quite dangerous to run down such a staircase! The formula looks like this:

$$R = \tfrac{1}{2}(24 - T)$$

R stands for Riser;
T stands for Tread.

Example:

Suppose we are building a house, and want the staircase tread to be 10 in wide. By *substituting* this number for T in the formula, we can work out what size is best for the riser. Instead of

$$R = \tfrac{1}{2}(24 - T)$$

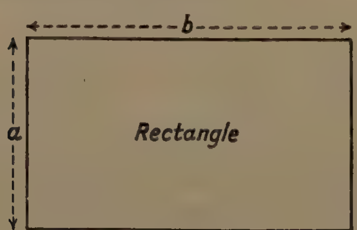
we now have $R = \tfrac{1}{2}(24 - 10)$

$$= \tfrac{1}{2} \times 14$$
$$= 7 \text{ in}$$

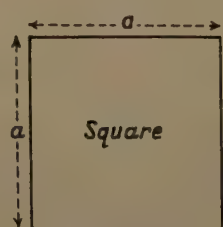
The riser must be 7 in high.

This is an interesting example of a formula which is only useful within certain limits. (If the tread is 24 in wide the formula doesn't work at all—the riser is 0!)

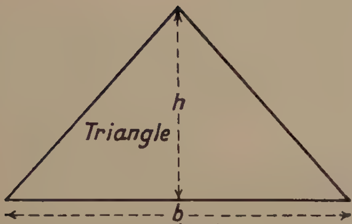
The areas of all the most common shapes can be worked out by formulae. Here are some of them:



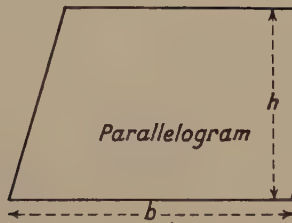
$A = ab$



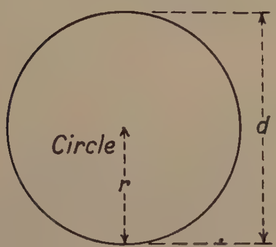
$A = a^2$



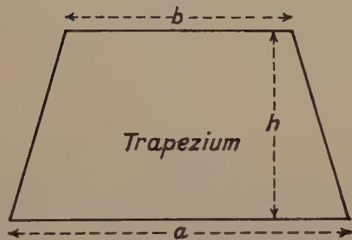
$A = \frac{1}{2}bh$



$A = bh$



$A = \pi r^2$ (π is about 3·14)



$A = \frac{1}{2}(a+b)h$

What are the areas of these shapes if

$$a = 3 \text{ cm}$$
$$b = 5 \text{ cm}$$
$$r = 4 \text{ cm}$$
$$\text{and } h = 6 \text{ cm?}$$

Don't forget that what a formula does is to give you instructions; it tells you what to do. Do you have a model aeroplane powered by rubber bands? Here is a formula which tells you how many times you can wind the propeller:

$$N = \frac{4L\sqrt{L}}{\sqrt{W}}$$

N is the number of turns.
L is the length of the rubber in inches.
W is the weight of each skein of rubber, in ounces.
Suppose $L = 9$
$W = 4$

Then N, the number of turns, is $\dfrac{4 \times 9 \times \sqrt{9}}{\sqrt{4}}$

$$= \frac{36 \times 3}{2}$$

$$= 54.$$

The propeller can be wound 54 times.

EXERCISE 2F

1. The R.A.C. formula for the horse-power rating of a car (which is *not* the brake horse-power) is $H = \frac{2}{5}D^2N$,

where $H =$ horse-power,
$D =$ diameter of each cylinder in inches,
$N =$ number of cylinders.

What is the R.A.C. rating of a car with 6 cylinders, each of 3 in diameter?

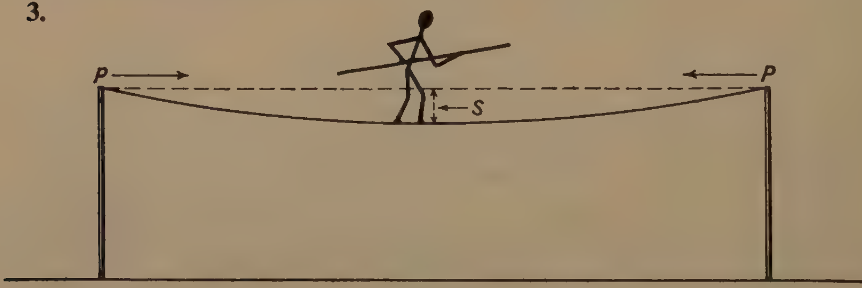
2. One formula used by insurance companies to work out the average time a man can expect to live is this:

$$E = \frac{2}{3}(80 - x),$$

where E is the expectation of life, in years.
x is the present age of the man, in years.

How much longer can a man expect to live if he is now 50 years old? If he is 77 years old? Using this formula, how long do you expect to live?

3.



The pull on the supports for this tight-rope is given by the formula $P = \dfrac{25WL}{S}$

where P = the pull, in kg
$W =$ the weight on the rope, in kg
$S =$ the sag, in cm
$L =$ the length of the rope, in m

How much pull is there on the supports of a tightrope which is 20 m long, sags 60 cm, and which is supporting an acrobat who weighs 60 kg?

4. The time a pendulum takes to swing is $\dfrac{11}{14}\sqrt{\dfrac{L}{2}}$ seconds, where L ft is the length of the pendulum. How long does an 8-ft pendulum take to swing?

5. Kilometres can be changed to miles using the formula: $M = \dfrac{5K}{8}$. How many miles are there between two towns 64 kilometres apart?

6. The weight that a plank will carry without breaking is given roughly by the formula $W = \dfrac{3bd^2}{4L}$,

where W = weight in cwt,
b = width of the plank in inches,
d = thickness of the wood in inches,
L = length of the plank in feet.

How many cwt will a plank 6 ft long, 1 in thick and 8 in wide carry? Would it be safe for a man to work on such a plank?

7. The breaking load in tons of a certain kind of rope is given by the formula $B = \dfrac{3}{5}C^2$

where B = breaking load in tons, and
C = circumference of the rope in inches.

Would it be safe to lift a load of $2\frac{1}{2}$ tons with a rope 2 in in circumference? How much could be lifted with such a rope?

8. $I = \dfrac{PRT}{100}$ gives the interest when money is invested,

where P = the amount invested,
R = the rate % per annum.
T = the time, in years.

How much interest would you receive if you invested £400 at 5% for 3 years?

9. Batting averages are worked out using the formula $\dfrac{r}{x - n}$,

where r = number of runs,
x = number of innings,
n = number of times not out.

What is the average of a cricketer who scores 350 runs in 9 innings, 2 times not out?

10. The weight of a small yacht is found by the formula $T = \dfrac{11\,(L - B)B^2}{2000}$

where T = the tonnage,
L = the length, in feet,
B = the breadth, in feet.

What is the weight of a yacht 30 ft long, 10 ft broad?

CHAPTER 3

EQUATIONS

YOU MAY have found that some parts of the last chapter were quite difficult at first, especially making your own formulae. This is because learning algebra is like learning a new language. In fact it *is* a new language, and it is when we begin to think in a language that we speak it well. Making formulae is an excellent way of learning to think algebraically. You won't find equations, the subject of this chapter, at all difficult—it is great fun, and in fact, you have already done some in Chapter One, perhaps without knowing it!

You can find out which of your friends are good at equations by asking them this question.

"I think of a number, double it, and add 4. The result is 10. What is the number I first thought of?" Of course you will have to work it out yourself first.

What would this question look like if we put it down in our algebraic shorthand?

First of all we would have to decide to use some letter for the unknown number. Let us call this one n. Now the question says the number is doubled

$n \times 2$, or $2n$,

and then 4 is added: $2n + 4$,

the result is 10: $2n + 4 = 10$.

What was the number I first thought of . . . $n = ?$

Now it is fairly easy to see that if twice the number ($2n$) with 4 *added* is 10, twice the number *without* the 4 is 6, or

$$2n = 6$$

If *twice* the number is 6, the number must be 3 or, in our shorthand,

$$n = 3.$$

Now $2n + 4 = 10$ is called an EQUATION, because all the terms on one side ($2n + 4$), are *equal* to those on the other (10). Both sides balance.

Notice the difference between an *expression*, for example $2a + b$, and an *equation* which has two equal sides.

It must have struck you that we have seen lots of equations when talking about formulae. $A = ab$ is an equation, and so is $R = \frac{1}{2}(24 - T)$. When we have worked out our unknown number, we are said to have *solved* the equation. See if you can solve these equations just by looking at them:

EXERCISE 3A

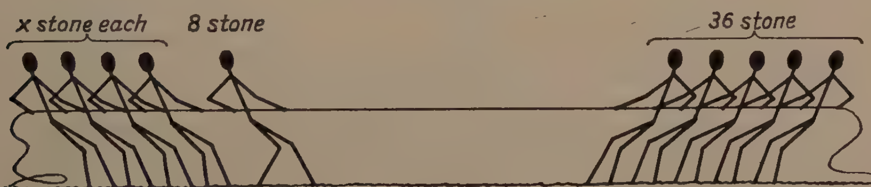
1. $a + 4 = 9$.
2. $n + 6 = 12$.
3. $2b = 16$.
4. $d + 3 = 3$.
5. $2x = 2\frac{1}{2}$.
6. $m + 2m = 90$.
7. $b - 5 = 1$.
8. $30 - n = 21$.
9. $8x = 24$.
10. $x - 1 = 0$.
11. $a + (a + 1) = 13$.
12. $n + (n - 1) = 13$.
13. $4(5x) = 40$.
14. $2(x + 3) = 14$.
15. $m(m - 1) = 12$.

Constructing an Equation

An equation is a sentence, written in shorthand. If we take the equation in No. 6 above, $m + 2m = 90$, how could we make an English sentence from it? We might say, "A boy is given a certain amount of pocket money, has twice that amount in his savings box and altogether he has 90p." Or we might say, "A cricketer scores a number of runs in one innings, and twice that number in his next. His total score for both innings was 90 runs." See if you can make up English sentences for other equations in this exercise.

Example:

Of course, we can do it the other way round too. Here is a sentence: "In a tug of war there were five boys on each side. Four boys on one side were all of the same weight plus one boy who weighed 8 stone. On the other side the total weight of the five boys was the same, and altogether that side weighed 36 stone.



Remembering that we use letters to stand for *numbers* that we don't know, or wish to find out, and *not* for quantities, we could say, "Let the four boys on the extreme left side weigh x stone each. Since there are 4 of them, their weight is $4x$ stone altogether. So the total weight of that side is

$$4x + 8 \text{ stone}$$

The boys on the right-hand side of the rope weigh 36 stone and their total weight is the same as (or equal to) the other side.

So, $4x + 8$ (on the left-hand side) $= 36$ stone (on the right), or

$$4x + 8 = 36$$

Here is our equation. Can you see what each boy on the left weighs? Using your own letters, can you make equations from these sentences? You may be able to solve them straight away. If not, it doesn't matter. We will work out a method later.

EXERCISE 3B

Make up a formula for each of following:

1. A man earns £6 a week, plus £4 for every refrigerator he sells. In one week he was paid £18. How many did he sell?

2. I think of a number, double it, and add 7 to it. The result is 13. What was the number?

3. Some airmen were chosen to become astronauts. Only a quarter of them passed the medical tests, and then 8 others were accepted. Altogether there were 12 in the final team. How many were first chosen?

4. If I add 15 to my young brother's age, the result is the same as multiplying his age by 4.

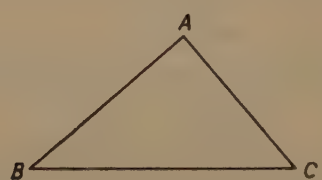
5. I think of a number, multiply it by 4, and add twice the number I first thought of. The answer is 24.

6. 20 p is taken out of a money box and what is left is equal to the original amount divided by 3.

7. What are the dimensions of a book that is twice as many inches long as it is wide, if its area is 18 in^2? (*Hint:* Let x in be the width.)

8. A jigsaw puzzle had 150 pieces when new and some were lost. When the owner counted the pieces left, he found he had 126.

9.



$$\hat{B} = 40°$$
$$\hat{C} = 50°$$

All the angles of a triangle add up to 180°. What is $\hat{A}$?

10. A car full of people weighs 20 cwt. The car weighs 4 times as much as the people. (Let the empty car weigh w cwt.)

Solving Equations

You have probably solved quite a number of the equations in this chapter already. The interesting thing is to try to remember *how* you did it. We ought to be able to find some rules which will enable us to solve all equations of this kind. Let's take an easy one first:

$$a + 4 = 9$$

You probably thought like this: If the unknown number has to have 4 added to it to equal 9, *without* the 4 it must be 5.

Let us write both of these statements in our shorthand:

$$a + 4 = 9$$
$$a = 5 \text{ (or } 9 - 4)$$

Can you see what has happened? We have taken the 4 from both sides of the equation. We could, in fact, have written:

$$a + 4 - 4 = 9 - 4$$

and, of course, *we must do the same thing* (in this case subtracting 4) *to both sides of the equation* if it is still to balance.

Let's try another:

$$y - 3 = 9$$

We want to end with $y = \ldots$ and this time we must *add* 3 (since $y - 3 + 3 = y$). So we must also add 3 to the other side,

$$y - 3 + 3 = 9 + 3$$
$$\therefore y = 12$$

(In case you have not come across it before $\therefore$ means therefore. What we are saying is that as $y - 3 + 3$ is equal to $9 + 3$, then, or therefore, y is equal to 12.)

As you would expect, the same rule, "do the same thing to each side," holds good for dividing and multiplying as well.

Example:

$$4x = 24$$

To get from $4x = \ldots$ to $x = \ldots$ we have to divide by 4. So both sides have to be divided,

$$\frac{4x}{4} = \frac{24}{4}$$

Cancelling, $$\frac{\cancel{4}x}{\cancel{4}} = \frac{\overset{6}{\cancel{24}}}{\cancel{4}}$$

$$\therefore x = 6$$

Example:

$$\frac{a}{3} = 9$$

To change $\frac{a}{3}$ (or one-third of a) to a, we must *multiply* by 3. Following the same rule, to make both sides balance, we multiply *both* by 3,

$$\frac{a}{3} \times 3 = 9 \times 3$$

Cancelling, $a = 27$

Sometimes two or more of these operations have to be done to solve the equation.

Example:

$$3b + 5 = 11$$
$$\therefore 3b + 5 - 5 = 11 - 5 \text{ (subtract 5 from both sides)}$$
$$\therefore 3b = 6$$
$$\therefore \frac{\cancel{3}b}{\cancel{3}} = \frac{\overset{2}{\cancel{6}}}{\cancel{3}} \text{ (divide both sides by 3 and cancel)}$$
$$\therefore b = 2$$

The interesting thing about equations is that you need never get them wrong! Let us check this one. If $b = 2$, then $3b + 5$ ought to equal 11. 3 times 2 is 6, plus 5 is 11, so 2 *is* the right number for b.

EXERCISE 3C

Find the value of x and check your answers:

1. $x - 4 = 8$.
2. $x - 17 = 0$.
3. $x - 429 = 500$.
4. $1 = x - 41$.
5. $x + 1 = 15$.
6. $x + 32 = 48$.
7. $x + 9 = 9$.
8. $5 = 2 + x$.
9. $3x = 33$.
10. $9x = 9$.
11. $8x = 3$.
12. $7 = 3x$.
13. $11x = 0$.
14. $\frac{x}{5} = 1$.
15. $\frac{x}{7} = 9$.
16. $\frac{x}{2} = 0$.
17. $x + \frac{3}{4} = 2$.
18. $\frac{x}{5} = \frac{1}{10}$.
19. $3x = 2\frac{1}{4}$.
20. $4x - 3 = 17$.

Position of the Unknown Quantity

There are several things to notice. One is that sometimes the unknown number is on the right, but this makes no difference. If you find that $3 = x$, that is the same as $x = 3$. Another is that there may be unknowns on both sides of the equation.

Example:

$$2x + 4 = x + 7$$
(Subtract 4 from both sides) $\therefore 2x = x + 7 - 4$
$$\therefore 2x = x + 3$$
(Subtract x from both sides) $\therefore 2x - x = x - x + 3$
$$\therefore x = 3$$

What we have done is to put all the unknowns on one side and all the numbers on the other.

You must have noticed by now that subtracting or adding to both sides is really moving the number to the other side, and changing its sign.

In this example the first line was

$$2x + 4 = x + 7$$

The next was $\quad 2x = x + 7 - 4.$

The 4 has been moved to the other side and the sign changed.

So we have, "*Change the side, change the sign,*" as a rule.

It must be said at once, however, it is no use learning this parrot fashion, if you don't remember what is really happening. You are adding or subtracting the same number from each side.

Here is one more example, and it shows one way of setting out your solution. Of course, the important thing is to understand what you are doing, but often a clearly set-out sum will help you to think clearly.

Example:

$7\frac{1}{2} = \frac{x}{2} + 5\frac{1}{2}$ (Write down the equation first, without any working.)

$\therefore 7\frac{1}{2} - 5\frac{1}{2} = \frac{x}{2}$ (Put all the numbers on one side, all the unknowns on the other. In this example, by subtracting $5\frac{1}{2}$ from both sides.)

$\therefore 2 = \frac{x}{2}$ (Collect the terms, that is in this case, by adding together the numbers. Sometimes you will have to add together the unknowns, if they are like terms on the other side.)

$\therefore 4 = x$ (or $x = 4$) (Multiply both sides by 2. *Half* the unknown is equal to 2, the *whole* must equal 4.)

Check: (Substitute 4 wherever x occurs in the *original* equation. The right-hand side adds up to $7\frac{1}{2}$, so the value for x is correct. When doing your check, do each side separately if there are unknowns on both sides.)

$\frac{4}{2} + 5\frac{1}{2} = 7\frac{1}{2}$

EXERCISE 3D

Solve these equations and check your answers.

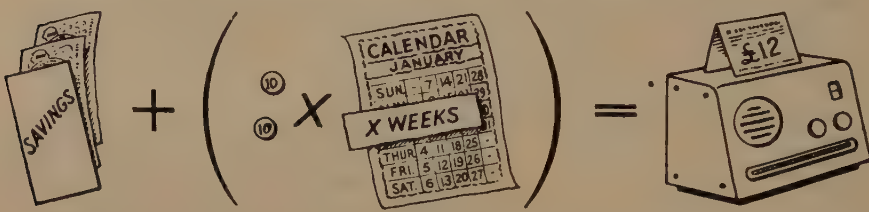
1. $47 = 23 + x.$
2. $11x = 121.$
3. $4x = 25.$
4. $7 = 11x.$
5. $4\frac{1}{2} = 3x.$
6. $3x + 3 = 6.$
7. $5x + 9 = 39.$
8. $7x + 3 = 66.$
9. $7x - 8 = 34.$
10. $85 = 12x + 1.$
11. $26 = 5x + 1.$
12. $2x + 3 = 11.$
13. $4x - 4 = 16.$
14. $\frac{x}{3} + 4 = 7.$
15. $\frac{1x}{5} - 1 = 0.$
16. $3\frac{1}{2} = \frac{x}{5} - 1\frac{1}{2}.$
17. $3x - 2 = x + 8.$
18. $7 + 3x - 3 = 5x.$
19. $\frac{1}{4}x + \frac{1}{2}x = 3.$
20. $\frac{x}{3} + 4x = 26.$

Problems

You have already had some practice in making equations from sentences (if you missed them you had better go back and do them now). Now that you can solve equations, you ought to be able to find the answers to many puzzles or problems.

Example:

I already have £2 in my Savings Book. If I save 20p in a week for x weeks I can buy a radio costing £12; for how long must I save?



The first thing to do is to read the question carefully and ask ourselves what sort of answer is required. In this case the key words are "how long." We have to find the time. We are told that this is x weeks. So we have to find x. The next thing to notice is that two of the amounts of money mentioned are in pounds and one in pence.

All the units in our equation should be in the same units of measurement and so we can either change the pounds to pence, or vice-versa. It doesn't matter which, but once we have decided we must stick to it. Let us use pence. Now we can compile our equation using all the information given in the question:

I have £2	200p
I save 20p a week for x weeks, total	$20x$p
I can buy a radio for £12	1200p
Our equation is	$200 + 20x = 1200$

Solve the equation (subtract 200 from each side)

$$200 + 20x - 200 = 1200 - 200$$
$$\therefore 20x = 1000$$
$$\therefore x = 50$$

Answer: I have to save for 50 weeks.

You may object that this is a long-winded way of doing something that you can probably do in your head very quickly. And so it is. But there are many problems which are difficult to do unless you understand this method.

Example:

When 45 is added to a number the answer is equal to the number multiplied by 6. Find the number.

Let x be the number	
45 is added to the number	$x + 45$
The number multiplied by 6	$6x$
They are equal	$x + 45 = 6x$
Subtract x from both sides	$\therefore 45 = 6x - x$
Collect the terms	$\therefore 45 = 5x$
Divide both sides by 5	$\therefore 9 = x$

The number is 9.

Check in the original question (not in your equation).

If the number is 9, $9 + 45 = 54$.

The number multiplied by $6 = 54$, so they *are* equal.

Example:

Here is a harder one. If you can understand how the equation is formed you will be able to do all the problems in this chapter.

A pet-shop owner buys some water-snails at 4p each and finds that 10 are dead. By selling the rest at 5p each he makes a profit of 75p. How many did he buy?

The question is "How many did he buy"	Let x water-snails be the number he bought
How much did he pay for them?	
Each one costs 4p, x must cost	$4x$ pence
How many does he sell?	$x - 10$
How much does he sell them for?	
Each one sells for 5p so $(x - 10)$ sell for	$5(x - 10)$ pence
So we have: Cost price	$4x$ pence
Selling price	$5(x - 10)$ pence
He makes a profit of	75 pence
If we *add* the profit to the cost price we shall have the selling price ..	$\therefore 4x + 75 =$ the selling price
But the selling price is	$5(x - 10)$ pence,
	$\therefore 4x + 75 = 5(x - 10)$

Here is our equation:

First remove the bracket (see Chap. Two)	$4x + 75 = 5x - 50$
Add 50 to both sides,	$\therefore 4x + 75 + 50 = 5x$
Subtract $4x$ from both sides,	$\therefore 75 + 50 = 5x - 4x$
Collect the terms,	$\therefore 125 = x$

Remember that the equation is only a means of solving the problem, and the solution of the equation is not the answer to the problem; i.e. the answer is *not* $x = 125$, but, "*He bought* 125 *water-snails.*"

Check:

He buys 125 snails at 4p each .. .. 500 pence
He sells (125 − 10) snails at 5p each (115 × 5) .. 575 pence
He makes 75p profit.
So that 125 water-snails *does* give the correct profit.

Summary:

To solve the problems:

1. Read the question carefully, deciding *what* you are required to find.
2. Use x (or a, or any letter you like), to stand for this number. (*But remember it is a number not a quantity.*) Don't for example, start a problem like this:

   Let x be the money required,
   or, Let x be the distance he ran.

   Instead, write,

   Let £x be the money required,
   Let x *miles* be the distance he ran.
3. By using the information in the question construct an equation. Make sure both sides *are* equal.
4. Make sure that the units are similar—i.e. don't use mixed units, pounds and pence, or hours and minutes, in the same equation.
5. Solve the equation.
6. Look back at the problem to see that you have answered the question asked (and not another of your own invention!).
7. Check in the original question.

EXERCISE 3E

Make equations from the following problems and solve them. In some you are given the first line.

1. I think of a number, multiply it by 3, and the result is 0. What is the number?

2. If I add 4 to twice a certain number, the result is the same as subtracting the number from 13.

3. A boy saved his pocket-money for 8 weeks, then put 50p of it in his savings box. When he counted the remainder, he found he had £1·10. How much did he receive every week? (Let x p be his weekly pocket-money.)

4. (a) If x is an integer (i.e. a whole number), what is the next integer above x?

(b) What is the next integer below x?

(c) An integer and the one above it together add up to 93. What is the integer? (Let x be the integer.)

5. (a) If x is an even number, what is the next even number above?

(b) What is the next even number below x?

(c) If three consecutive even numbers total 96, what is the *lowest* even number?

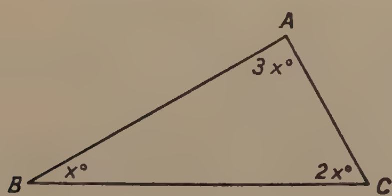
6. 4 times a number is equal to 19 minus 3 times the same number. What is the number?

7. A car park attendant noticed that if he halved the number of cars parked, and added 8, he had the same number as were parked on the previous day when he had counted 36. How many cars were parked?

(Let x cars be the number parked.)

8. Two brothers wanted to buy a football which cost 75p. The elder brother put 60p towards it, and the younger put a third of his monthly pocket-money. How much pocket-money did he get a month?
(Let xp be the younger brother's pocket-money.)

9.



(a) What is x? (*Hint:* The angles of a triangle add up to 180°.)

(b) What kind of angle is A?

10. A rectangular carpet is 12 ft wide and measures 66 ft all round. How long is it? (Let x ft be the length of the carpet.)

11. Jill bought 4 lb of oranges and 3 lb of grapes. The grapes were twice as expensive per pound as the oranges. In all she spent 40p. How much per pound were the oranges?

(Let x pence be the cost per pound of oranges.)

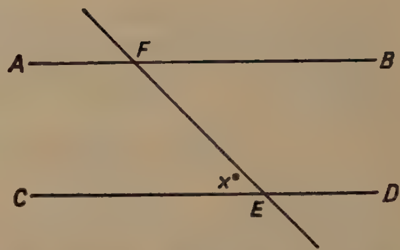
12. What number is greater than 8 by the same amount that it is less than 90?

13. Joe was a botanist as well as a mathematician. He had two plants. One was 12 cm tall and grew at the rate of 2 cm a week. The other was 3 cm tall and grew at the rate of 5 cm a week. Joe was able to calculate how many weeks it would be before they were both the same height. Can you?

(Let x weeks be the time they have to grow to reach the same height.)

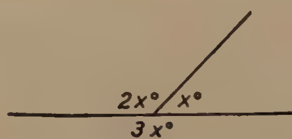
14. Young William was a prodigy. His father was 32 years old, and William knew that if he added three-quarters of his own age to that, he would get the same number as multiplying his own age by 4 and adding 6 years. How old was William? (Let x years be William's age.)

15. (a) How big is $B\hat{F}E$?
(See page 301, properties of parallels.)
(b) How big is $A\hat{F}E$?
(c) If $A\hat{F}E$ is $3x°$, what is x?



16. If I add 19 to a number, and multiply the result by 2, I get 126. What is the number?

17. How big is x?



What is x?
(*Hint:* How many degrees are there altogether?)

18. Paul had 13 marks less than Luke in an exam. When they added their marks together, they found the sum to be 187. What were their marks? (Let x marks be Luke's exam. score.)

19. The formula for changing Fahrenheit to Centigrade temperatures is $F = \frac{9C}{5} + 32$, where F is the Fahrenheit and C the Centigrade temperature. Find the Centigrade temperature which corresponds to 86° Fahrenheit.

20. The circumference of a circle can be found by the formula $C = 2\pi r$ where C is the circumference, r is the radius, and π ("pi") is approximately 3·14. What is the radius of a circle whose circumference is 18·84 in?

Symbols Used to Represent Inequality

In this chapter we have been talking about a special kind of mathematical sentences, and because both sides are equal, we call these equations. But there are many other sentences in which both sides are not equal, and there are some algebraic signs for these:

$\neq$ means "is *not* equal to,"
$>$ means "is greater than,"
$<$ means "is less than."

Example:

(a) $12 \neq 8 + 5$,
(b) $8 > 7$,
(c) $2 + 3 < 6$.

It is easy to remember in (b) and (c) which is which. The arrow always points to the smaller number. Can you guess what $\nless$ means, and $\ngtr$?

EXERCISE 3F

Which of the following statements are true, and which false?

1. $5 + 1 < 2 + 3$.
2. $7 + 3 < 3 + 7$.
3. $3 \times 0 > 3$.
4. $4 + 7 \neq 7 + 2$.
5. $4 - 2{\cdot}5 > 1{\cdot}4$.
6. $6\frac{1}{2} - 2\frac{1}{4} < 5\frac{1}{4}$.
7. $2(4 + 7) > 3(4 + 3)$.
8. $\frac{1}{3} + \frac{2}{5} \neq \frac{9}{15}$.
9. $8 + 3 \neq 4 + 7$.
10. $4 + x \neq 4x$.

True or False?

Look at this sentence: $x + 3 < 6$.
Can we find any values for x which would make this true?

Substituting,	If $x = 0$,	$0 + 3 < 6$	TRUE
	If $x = 1\frac{1}{4}$,	$1\frac{1}{4} + 3 < 6$	TRUE
	If $x = 2$,	$2 + 3 < 6$	TRUE
	If $x = 2\frac{1}{2}$,	$2\frac{1}{2} + 3 < 6$	TRUE
	If $x = 3$,	$3 + 3 < 6$	FALSE
	If $x = 3\frac{1}{2}$,	$3\frac{1}{2} + 3 < 6$	FALSE
	If $x = 4$,	$4 + 3 < 6$	FALSE
	If $x = 5$,	$5 + 3 < 6$	FALSE

So x can be any value less than three.

This is the first example you have seen of a sentence in which several values for an unknown can make the statement true. What values for x make the following statements true?

EXERCISE 3G

1. $x + 7 < 9$.
2. $10 - x < 4$.
3. $x + 5 > 9$.
4. $x - 5 > 9$.
5. $x - 0 > 4$.
6. $2x < 8$.
7. $\frac{x}{2} > 5$.
8. $3x \neq 9$.
9. $2(x + 5) \neq 18$.
10. $\frac{x}{3} \neq 12$.

CHAPTER 4

INDICES AND FRACTIONS

YOU REMEMBER that in Chapter 1 we used a shorthand method of writing numbers which were multiplied by themselves.

Example:

7×7 was called 7^2 ("seven squared"),
$a \times a = a^2$ ("a squared"),
$B \times B = B^2$, and so on.

The small ^{2} is called an INDEX number (plural, indices) or EXPONENT.

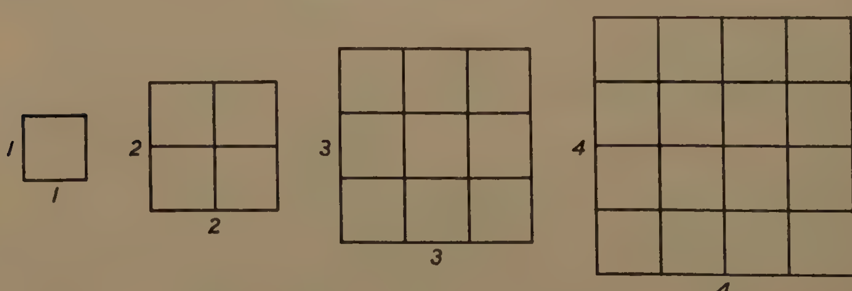
When a number is multiplied by itself three times it is said to be "cubed." Thus

$$x \times x \times x = x^3 \text{ ("}x\text{ cubed")}$$

If a number is multiplied by itself more than three times, the term "to the power" is used. Thus

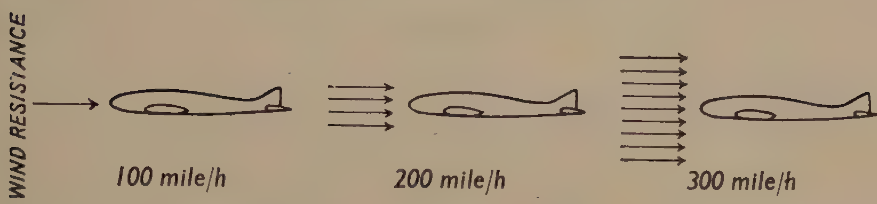
$$y \times y \times y \times y = y^4 \text{ ("}y\text{ to the power 4")}$$
$$1 \times 1 \times 1 \times 1 \times 1 = 1^5 \text{ ("1 to the power 5")}$$

You see that the index number is used to show how many times a number is multiplied by itself. This is an immensely convenient way of writing large numbers. 10^7 is much quicker than 10,000,000! Squares and cubes (and indeed other powers) are connected with the way in which things grow, and it is easy to see from this diagram why a number multiplied by itself is usually called the square of the number.



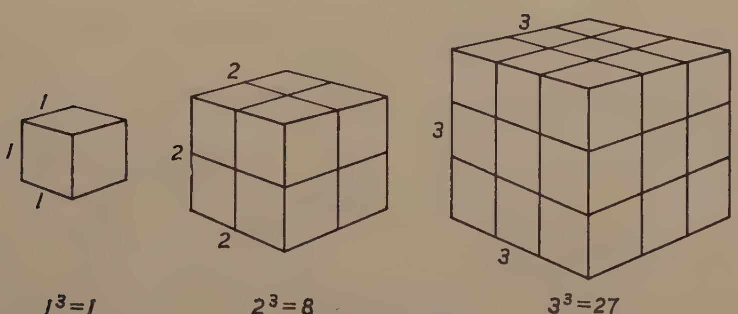
$1^2=1$ $2^2=4$ $3^2=9$ $4^2=16$

If the speed of an aeroplane is doubled, the air resistance increases 4 times; if it is trebled, the resistance increases 9 times:



The speed grows 1, 2 and 3, the resistance grows 1, 4, 9.

This is an example of the square law, and there are many things in common use, the amount of weight a rope or a beam will hold, for example, or the force exerted by the wind blowing at different speeds, which depend on it. Similarly the name "a cubed" for "a to the power 3" stems from the way in which the volume of a cube grows:



If every dimension of the cube is doubled, the volume increases 8 times. If every dimension of the cube is trebled, the volume increases 27 times. You remember, too, from Chapter 1, that

$$2x^2 \text{ means } 2 \times x^2, \text{ or } 2 \times x \times x$$
$$\text{and } 4y^3 \text{ means } 4 \times y \times y \times y.$$

The number part of these expressions is called the COEFFICIENT.

2 is the coefficient of x^2 in $2x^2$
4 is the coefficient of y^3 in $4y^3$
5 is the coefficient of a^3y in $5a^3y$
1 is the coefficient of x^2 in x^2 (there is only *one* x^2)
1 is the coefficient of y in the expression $3x + y$

x, x^2 and x^3 are all different *powers* of x and are unlike terms, so cannot be added together (at least, not until we know the value of x).

But we can multiply or divide them.

Let us see what happens:

Multiplication

Example: $x^3 \times x^2$.

Remember that x^3 is shorthand for $x \times x \times x$

$$\text{and } x^2 = x \times x$$

$$\therefore x^3 \times x^2 = x \times x \times x \times x \times x$$
$$= x^5$$

Example: $x^2 \times x$

$$x^2 = x \times x$$
$$\therefore x^2 \times x = x \times x \times x$$
$$= x^3$$

When powers of the same number are multiplied, ADD the indices.

But they must be powers of the *same* number:

$$2^2 \times 2^2 = 2^4$$
$$= 16$$

But, $2^2 \times 3^2 = 36$, *not* 6^4, which is 1,296!

So that, $x^3 \times y^4 = x^3y^4$,

and that is as far as we can go. (Once again, until we know what numbers x and y stand for.)

Example: $3a^2 \times 5a^4$

$$= 3 \times 5 \times a^2 \times a^4$$
$$= 15a^6$$

Example: $2m^3 \times 7a^2$

$$= 2 \times 7 \times m^3 \times a^2$$

$= 14a^2m^3$ (The numbers can be multiplied, and the unknowns are usually put in alphabetical order.)

Division

Example: $a^5 \div a^2$

We can also write it this way: $\dfrac{a^5}{a^2}$

which is, expanded, $\dfrac{a \times a \times a \times a \times a}{a \times a}$

Since there are multiplication signs only in this fraction, we can cancel;

$$\frac{\cancel{a} \times \cancel{a} \times a \times a \times a}{\cancel{a} \times \cancel{a}}$$

$$= a \times a \times a$$
$$= a^3$$

$\therefore a^5 \div a^2$ gives a^3.

For division, SUBTRACT the indices.

Once again, they must be powers of the *same* number (in this case, a).

Example:

$$\frac{6m^5}{3m^3}$$

$$= \frac{\overset{2}{\cancel{6}} \times \cancel{m} \times \cancel{m} \times \cancel{m} \times m \times m}{\cancel{3} \times \cancel{m} \times \cancel{m} \times \cancel{m}}$$

$$= 2 \times m \times m$$

$$= 2m^2$$

Example:

$$\frac{12ab^2}{4b^2a}$$

$$= \frac{\overset{3}{\cancel{12}} \times \cancel{a} \times \cancel{b} \times \cancel{b}}{\cancel{4} \times \cancel{b} \times \cancel{b} \times \cancel{a}}$$

$= 3$ (Since ab^2 means $a \times b \times b$, and b^2a means $b \times b \times a$, they are only different ways of writing the same thing.)

Squares and Square Roots

Example: Simplify $(2x)^2$ (Say: "$2x$ all squared.")

This means: $2x \times 2x$

$= 4x^2$ Notice how $(2x)^2$ is different from $2x^2$.

Example: Simplify $\sqrt{49x^4}$.

When we look for the square root of say, 81, we are really looking for the number, which when multiplied by itself, gives 81, in this case 9.

What number multiplied by itself gives $49x^4$? It can't be $7x^4$, because $7x^4 \times 7x^4$ would equal $49x^8$. So not only do we have to find the square root of 49, but also of x^4. In expanded form

$$x^4 = x \times x \times x \times x.$$

This can be split into two equal parts or factors

$$(x \times x) \times (x \times x)$$
$$\text{or } x^2 \times x^2$$

which when multiplied together give x^4. So we could factorize $49x^4$ like this:

$$(7 \times 7) \times (x^2 \times x^2)$$
$$\text{or } (7 \times x^2) \times (7 \times x^2)$$
$$\therefore \sqrt{49x^4} = \sqrt{7x^2 \times 7x^2}$$
$$= 7x^2$$

Check: $7x^2 \times 7x^2$ does give $49x^4$,

$\therefore$ $7x^2$ is the square root of $49x^4$.

(In Chapter 4 you will learn that there is another possible answer.)

EXERCISE 4A

Simplify the following expressions. For No. 1-10 write out each expression in full before you multiply or cancel.

1. $m^2 \times m^2$

2. $a^4 \times a$

3. $2b^2 \times 3b$

4. $c^5 \div c^4$

5. $e^{10} \div e^5$

6. $4f^2 \div 2f$

7. $(3x)^2$

8. $(2x^2)^3$

9. $\sqrt{x^6}$

10. $\sqrt{9y^4}$

11. $g^2 \times g^5$

12. $h \times h^9$

13. $k^2 \times 5k^2$

14. $e^4 \times 3e$

15. $5m^2 \times 25m^7$

16. $3p^3 \times 3p^3$

17. $2a^2 \times b$

18. $2a^2 \times ab$

19. $5r^4 \times s^2$

20. $t^3 \div t^2$

21. $u^2 \div u$

22. $v^4 \div v^4$ (Write this one out in full.)

23. $w^8 \div w^4$

24. $2y^2 \div y$

25. $\dfrac{4z^2}{z^2}$

26. $\dfrac{x^3}{4x^3}$

27. $\dfrac{a^2}{5a}$

28. $\dfrac{15a^2}{5a^3}$

29. $\dfrac{b^2 \times 2b^2}{b^3}$

30. $\dfrac{c \times c^2}{c^4}$

31. $(a^3)^2$

32. $(3d^2)^3$

33. $(3d^3)^2$

34. $(2e)^4$

35. $\dfrac{(2x)^2}{x^2}$

36. $\sqrt{4x^2}$

37. $\sqrt{36y^6}$

38. $\sqrt{9x^4}$

39. $\sqrt{a^2b^2}$

40. $\sqrt{c^2d^4}$

41. $\sqrt{e^6f^8}$

42. $(9gh^2)^2$

43. $\dfrac{12kl}{4k^2l^2}$

44. $\dfrac{2a^3 \times 4}{6a}$

45. $\dfrac{k^8}{k \times k^4}$

46. $(2p)^2 \div 2p^3$

47. $\frac{1}{2}x \times \frac{1}{2}xy$

48. $\dfrac{4x}{4}$

49. $\dfrac{8ab}{\sqrt{16b^2}}$

50. $\dfrac{(2m)^2 \times 4n}{8mn}$

H.C.F. and L.C.M.

You must have met this sort of problem in arithmetic:

The top of a kitchen table is 4 ft long and 3 ft 6 in wide, and I wish to cover the top with square tiles. What is the biggest size I can use as a complete square, without having to cut any of them?

The table is 48 in long and 42 in wide. The question really being asked is "What is the largest number which will divide both 48 and 42 exactly?" In other words, the highest common factor, or H.C.F. (You can probably guess the answer.)

You have used L.C.M's already when adding fractions. For example;

$$\frac{2}{5}+\frac{4}{7}=\frac{14+20}{35}=\frac{34}{35},$$

where 35 is the L.C.M. of 5 and 7.

Problems similar to these occur in algebra, and it is useful to be able to find the H.C.F. or L.C.M. of groups of algebraic expressions. But it is very easy to become muddled about the difference between H.C.F.s and L.C.M.s, so it is just as well to remind ourselves exactly what they mean.

Highest Common Factors

6 divides into 36 exactly, so 6 is a *factor* of 36. (So are 1, 2, 3, 4, 9, 12 and 18, as well as 36 itself.)

A *prime* number is divisible only by itself and 1 (11 is a prime number and so is 19).

A *prime factor* is both a factor and a prime number.

6 is a factor of 36, but not a prime number.

3 is a factor of 36 and is a prime number, so is a prime factor. (What are the other *prime* factors of 36?)

We can express a number as a product of prime factors in only one way. Consider the two numbers 42 and 63. The prime factors of 42 are $2 \times 3 \times 7$ (they are all prime numbers and their product is 42).

The prime factors of 63 are $3 \times 3 \times 7$

or $3^2 \times 7$

Have these two numbers any factors in common (i.e. shared)?

7 is one common factor, and so is 3.

So they share 3×7, or 21, and 21 is the *Highest Common Factor*, or H.C.F.

Example:

What is the H.C.F. of 40 and 60?

The prime factors of 40 are $2^3 \times 5$

and of 60 are $2^2 \times 3 \times 5$

$\therefore$ The Common Factors are 2^2 and 5

and the H.C.F. is $2^2 \times 5 = 20$

Finding the H.C.F. of numbers with unknowns is done in exactly the same way:

Example:

Find the H.C.F. of $12x^2y$ and $20xy^2$.

The prime factors of $12x^2y$

are $2^2 \times 3 \times x^2 \times y$

The prime factors of $20xy^2$

are $2^2 \times 5 \times x \times y^2$

$\therefore$ The common factors are 2^2, x and y (Notice that x is common to both, but not x^2, and similarly y is common to both, but not y^2.)

$\therefore$ The H.C.F. is $2^2 \times x \times y$ or $4xy$

EXERCISE 4B

Find the H.C.F. of:

1. $2x$, $4x$
2. $4a$, $6a$
3. $9b$, $3b^2$
4. $3c^2$, $2c^3$
5. d, efg (There *is* an answer!)
6. $12g^2h$, $4hk^2$
7. $3x^3y^4$, $12xy^2$, $9x^2y^3$
8. $8r^2st$, $18r^3s^3t^3$
9. $5a^2b^2c^2$, $10a^2b^2c^2$
10. $22xy^3$, $11xy^4$

Lowest (or Least) Common Multiples

Any number which can be divided by another is called a *multiple* of the second. 54 is a multiple of 9 (and of 6, and 2, and 3, since they all divide exactly into 54), and 4 is a multiple of 2 (and of 1, *and* 4, since 4 divides into itself exactly).

Consider the two numbers 6 and 8.

The multiples of 6 are, 6, 12, 18, 24, 30, 36, 42, 48 . . . etc.
The multiples of 8 are, 8, 16, 24, 32, 40, 48 . . . etc.
Two of these multiples are common to both numbers, 24, and 48.
There are lots of others. Can you think of some?
But the *lowest* multiple common to both is 24, or the L.C.M. is 24.

Example.

What is the L.C.M. of x^2y^3 and x^4y?

The factors of x^2y^3 are $x \times x \times y \times y \times y$.
The factors of x^4y are $x \times x \times x \times x \times y$.

A multiple which both will divide into must contain at least $x \times x \times x \times x$ and $y \times y \times y$, or $x^4 \times y^3$.

$\therefore$ The L.C.M. is x^4y^3.

The L.C.M. contains the highest power of each factor.

(Can you say how many times x^2y^3 is contained in the L.C.M.?)

Example:

What is the L.C.M. of $12a^2bc$ and $45a^3bc^3$?

$$12a^2bc = 2^2 \times 3 \times a^2 \times b \times c$$
$$45a^3bc^3 = 3^2 \times 5 \times a^3 \times b \times c^3$$
$$\therefore \text{The L.C.M. is } 2^2 \times 3^2 \times 5 \times a^3 \times b \times c^3$$
$$\text{or } 180a^3bc^3$$

How many times is $12a^2bc$ contained in the L.C.M.?

$$\frac{\overset{15ac^2}{\cancel{180a^3bc^3}}}{\cancel{12a^2bc}}$$
$$= 15ac^2 \text{ times.}$$

EXERCISE 4C

Find the L.C.M. of:

1. $2^4 \times 3 \times 5^2$, $2^3 \times 3^2 \times 5^2$
2. $2x$, $6x^2$
3. $5y$, $4xy^3$
4. $2m^2$, $8mn^2$
5. $5r$, $125rs$
6. x^3, $18x^2y^4$
7. $3a^2$, $12b^2$
8. $5x^2$, $7y^4$
9. $3x^2$, $4x^3$, $5x^4$
10. $2x^2$, $4xy$, y^2

Find both the H.C.F. and L.C.M. of the following expressions. Be quite sure you know which is which. If in doubt remember what H.C.F. stands for and what a factor is: what L.C.M. stands for and what a multiple is.

11. a, a^2, a^3
12. $11x^2y$, $121y$
13. $3b^2cd$, $4bc^2d$
14. $27e^2f$, $3ef^5$
15. $7gx^2$, $13a^2b$
16. $(2a)^2$, $6ab$, $(2b)^3$

Fractions

When dealing with a fraction like $\frac{25}{40}$, you have learnt to simplify it to $\frac{5}{8}$ by cancelling. In other words by dividing both numerator and denominator by 5. In this case 5 is the Highest Common Factor of 25 and 40. $\frac{48}{60}$ becomes $\frac{4}{5}$, and 12 is the H.C.F. of 48 and 60. The same rule applies when simplifying algebraic fractions. Consider this fraction: $\dfrac{x^3}{x}$

Written out in full it becomes: $\dfrac{x \times x \times x}{x}$

Divide both numerator and denominator by x,

$$= \frac{\overset{1}{\cancel{x}} \times x \times x}{\underset{1}{\cancel{x}}}$$

$$= \frac{1 \times x \times x}{1} = \frac{x^2}{1}, \text{ or } x^2$$

Example:

Simplify $\dfrac{2x^2y^3}{8xy^4}$

The factors of $2x^2y^3$ are $2 \times x \times x \times y \times y \times y$.
The factors of $8xy^4$ are $2 \times 2 \times 2 \times x \times y \times y \times y \times y$.
$\therefore$ The H.C.F. of the two numbers is $2xy^3$.

$$2x^2y^3 \div 2xy^3 = x$$
$$8xy^4 \div 2xy^3 = 4y$$

$\therefore$ The fraction is simplified to $\dfrac{x}{4y}$

Perhaps an easier method is to do this:

Step 1: $\dfrac{\overset{1}{\cancel{2}}x^2y^3}{\underset{4}{\cancel{8}}xy^4}$ (Divide numerator and denominator by 2.)

$$= \frac{x^2y^3}{4xy^4}$$

Step 2: $\dfrac{\overset{x}{\cancel{x^2}}y^3}{4\underset{1}{\cancel{x}}y^4}$ (Divide both by x.) $= \dfrac{xy^3}{4y^4}$

Step 3: $\dfrac{x\cancel{y^3}}{4\underset{y}{\cancel{y^4}}}$ (Divide both by y^3) $= \dfrac{x}{4y}$

In other words, cancel the numeral part of the expression first, if possible, then each unknown in turn. All of these operations can be done together, provided you are quite clear about what is happening, and the calculation would then look like this:

$$\frac{\cancel{2}\overset{x}{\cancel{x^2}}\cancel{y^3}}{\underset{4}{\cancel{8}}\cancel{x}\underset{y}{\cancel{y^4}}} = \frac{x}{4y}$$

EXERCISE 4D

Simplify these fractions.

1. $\dfrac{2x}{2y}$
2. $\dfrac{5a}{10b}$
3. $\dfrac{12x}{16z}$
4. $\dfrac{3m}{3m}$ (The answer is *not* 0.)
5. $\dfrac{7c}{5c^2}$
6. $\dfrac{3r^2}{r}$
7. $\dfrac{s^3}{s^2}$
8. $\dfrac{t^8}{t^2}$
9. $\dfrac{3v}{5v}$
10. $\dfrac{4a^3}{a}$
11. $\dfrac{3ab^2}{a}$
12. $\dfrac{11c^2d}{2cd}$
13. $\dfrac{2c^2f^2}{2cf}$
14. $\dfrac{24x^2y^2}{20y^2}$
15. $\dfrac{3xy}{9x^2y^2}$
16. $\dfrac{12ef}{48}$
17. $\dfrac{k^6}{k^{12}}$
18. $\dfrac{8a^2b^5}{6a^3b}$
19. $\dfrac{3x^5y^5}{3x^5y^4}$
20. $\dfrac{2ab}{4ba}$ (Is $a \times b$ the same as $b \times a$?)
21. $\dfrac{x^5}{5x^2}$

22. $\dfrac{x^8}{x^4}$

23. $\dfrac{x^2y^2z}{xyz}$

24. $\dfrac{3a^2bc^3}{ab^2c}$

25. $\dfrac{18xy}{54x^4y^4}$

26. $\dfrac{15m^2nx^3}{6m^2}$

27. $\dfrac{21ab^2}{24a^2b^3}$

28. $\dfrac{14x^4y^6}{7x^2y^3}$

29. $\dfrac{15a^2x}{45xy^2}$

30. $\dfrac{x^3y}{(2xy)^2}$ (Work out the square first.)

Multiplication

You remember that with numerical fractions, multiplication consists of multiplying the numerators by each other, and then the denominators, cancelling first where possible. Thus:

$$\frac{7}{15} \times \frac{4}{5} = \frac{28}{75}$$

$$\text{and } \frac{11}{\overset{}{\underset{3}{\cancel{15}}}} \times \frac{\overset{1}{\cancel{5}}}{7} = \frac{11}{21}$$

The same rules apply for algebraic fractions.

Example: $\dfrac{2x^2}{y} \times \dfrac{5x^2}{3y^2} = \dfrac{2x^2 \times 5x^2}{y \times 3y^2}$

In expanded form this is $\dfrac{2 \times x \times x \times 5 \times x \times x}{y \times 3 \times y \times y}$

$$= \frac{2 \times 5 \times x \times x \times x \times x}{3 \times y \times y \times y}$$

$$= \frac{10x^4}{3y^3}$$

Example: $\dfrac{12x^3}{5y^2} \times \dfrac{15y}{4x} = \dfrac{\overset{3}{\cancel{12}} \times \cancel{x} \times x \times x \times \overset{3}{\cancel{15}} \times \cancel{y}}{\cancel{5} \times \cancel{y} \times y \times \cancel{4} \times \cancel{x}}$

$$= \frac{9x^2}{y}$$

Once you understand the method, it is not necessary to write out the fractions in expanded form, and this example would then look like this:

$$\frac{\overset{3x^2}{\cancel{12x^3}}}{\underset{y}{\cancel{5y^2}}} \times \frac{\overset{3}{\cancel{15y}}}{\cancel{4x}} = \frac{9x^2}{y}$$

EXERCISE 4E

Simplify the following:

1. $b^2 \times b$
2. $2a \times a^5$
3. $\frac{a^2}{b} \times \frac{a}{b^5}$
4. $\frac{2a^2}{3b} \times \frac{a^9}{5b^5}$
5. $\frac{4a}{3c^2} \times \frac{b^2}{5d}$
6. $\frac{4x}{3y} \times \frac{3x^3}{2y^3}$
7. $\frac{5x}{2y} \times \frac{7y^2}{x^2}$
8. $\frac{2x^2}{3y} \times \frac{6y^2}{4x}$
9. $\frac{15x^5}{12a} \times \frac{4a^4}{5x^6}$
10. $\frac{21ax}{7xy^3} \times \frac{4ay^4}{24a^2b}$

Division

In the same way, dividing algebraic fractions follows the same process used with numerical fractions. Thus:

$$\frac{7}{15} \div \frac{4}{5} = \frac{7}{\cancel{15}_{3}} \times \frac{\cancel{5}^{1}}{4}$$ (To divide by a fraction, turn it upside down and multiply.)

$$= \frac{7}{12}$$

and $$\frac{2a^2x}{3b^2x} \div \frac{4ax}{5bx}$$

$$= \frac{2a^2x}{3b^2x} \times \frac{5bx}{4ax}$$

$$= \frac{\cancel{2a^2x}^{a}}{\cancel{3b^2x}_{b}} \times \frac{5\cancel{bx}}{\cancel{4ax}_{2}}$$

$$= \frac{5a}{6b}$$

EXERCISE 4F

Simplify the following:

1. $\frac{7}{9} \div 3$
2. $5 \div \frac{5}{7}$
3. $\frac{a}{b} \div c$
4. $x \div \frac{x}{y}$
5. $2a \div \frac{2b}{2a}$
6. $\frac{1}{m} \div p.$
7. $\frac{1}{3a^2} \div \frac{1}{3a^2}$
8. $\frac{2e^3f}{3e^3f^2} \div \frac{6e}{4ef^4}$
9. $\frac{18xy^2}{3xy} \div \frac{27xz}{18yz}$
10. $\frac{4a^2b}{5a^3b} \times \frac{30a}{8b} \div \frac{6b}{12b}$

Addition and Subtraction

A man leaves $\frac{2}{5}$ of his money to his son, $\frac{3}{8}$ to his daughter, and the rest to a home for stray dogs. What fraction of his money goes to the dog's home?

To solve this problem we have to add together the fractions of his money that he has given to his children, and subtract this sum from the whole of it. It would then look like this:

$$\begin{aligned} 1 - (\tfrac{2}{5} + \tfrac{3}{8}) &= 1 - (\tfrac{16}{40} + \tfrac{15}{40}) \\ &= 1 - \tfrac{31}{40} \\ &= \tfrac{9}{40} \end{aligned}$$

This has meant carrying out three operations:

Step 1: Finding an L.C.M. for the denominators (in this case 5 and 8).

Step 2: Changing each fraction to an equivalent fraction, with the L.C.M. as the new denominator in each case.

Step 3: Adding or subtracting these fractions.

You have already practised finding L.C.M's (Step 1). Let us try Step 2.

Example:

Complete the following:

(a) $\frac{3}{5} = \frac{****}{25}$ (b) $\frac{r}{5} = \frac{****}{30}$ (c) $\frac{3}{x} = \frac{****}{x^2}$

(a) The second denominator is multiplied by 5, so the numerator also must be 5 times greater, if the two fractions are to be equal,

$$\therefore \frac{3}{5} = \frac{3 \times 5}{25} = \frac{15}{25}$$

(b) The denominator is 6 times greater, so the numerator must be multiplied by 6,

$$\therefore \frac{r}{5} = \frac{6r}{30}$$

(c) The denominator is multiplied by x, so the numerator is multiplied by x,

$$\therefore \frac{3}{x} = \frac{3x}{x^2}$$

EXERCISE 4G

Complete the following:

1. $\frac{1}{4} = \frac{***}{12} = \frac{***}{44}$
2. $\frac{3}{5} = \frac{9}{***} = \frac{27}{***}$
3. $\frac{x}{y} = \frac{***}{2y}$
4. $\frac{a}{b} = \frac{***}{bc^2}$
5. $\frac{e}{f} = \frac{***}{f^3}$
6. $\frac{2x}{a} = \frac{***}{ax}$
7. $\frac{y}{x} = \frac{***}{2x^2}$
8. $\frac{a}{2m} = \frac{***}{8mn}$
9. $\frac{5x}{ay} = \frac{30x^4}{***}$
10. $\frac{3p}{5m} = \frac{6p^3}{***}$

Now we are ready for Step 3.

Example:
$$\frac{3x}{5}+\frac{2x}{3}$$

Step 1: L.C.M. of 5 and 3 is 15.

Step 2:
$$\frac{3x}{5}=\frac{9x}{15} \text{ and } \frac{2x}{3}=\frac{10x}{15}$$

Step 3:
$$\therefore \frac{3x}{5}+\frac{2x}{3}=\frac{9x}{15}+\frac{10x}{15}$$

The denominator is common, so we can write
$$\frac{9x+10x}{15}$$
$$=\frac{19x}{15}$$ (9x and 10x are like terms and so can be added)

Example:
$$\frac{x}{2}+\frac{x}{4y}$$

Step 1: The L.C.M. of 2 and $4y$ is $4y$.

Step 2: $\frac{x}{2}=\frac{2xy}{4y}$ (multiplying by $2y$),

$\frac{x}{4y}$ already has the denominator $4y$,

Step 3:
$$\therefore \frac{x}{2}+\frac{x}{4y}=\frac{2xy}{4y}+\frac{x}{4y}$$

$2xy$ and x are *unlike* terms and so cannot be added,

so that $\frac{2xy+x}{4y}$ is as far as we can go.

Example:
$1-\frac{c}{d}$ (which equals $\frac{1}{1}-\frac{c}{d}$

Step 1: L.C.M. of 1 and d is d.

Step 2: $\frac{1}{1}=\frac{d}{d}$ and $\frac{c}{d}$ already has the common denominator.

Step 3:
$$\therefore 1-\frac{c}{d}=\frac{d}{d}-\frac{c}{d}$$
$$=\frac{d-c}{d}$$ (c and d are unlike terms and cannot be added).

(Remember that a fraction like this, $\frac{5-3}{5}$, *cannot be cancelled*, but this, $\frac{5\times 3}{5}$ can. Why?)

Example:

$$x - \frac{y}{z} = \frac{x}{1} - \frac{y}{z}$$

L.C.M. of 1 and z is z.

$$\therefore \frac{x}{1} = \frac{xz}{z},$$

and $\frac{y}{z}$ already has the common denominator.

$$\therefore \frac{xz}{z} \times \frac{y}{z} = \frac{xz + y}{z}$$

EXERCISE 4H

Simplify the following:

1. $\frac{a}{2} + \frac{a}{5}$ (Which is another way of saying $\frac{1a}{2} + \frac{1a}{5}$)
2. $\frac{a}{3} + \frac{a}{7}$
3. $\frac{x}{3} + \frac{2x}{9}$
4. $\frac{3x}{4} + \frac{2x}{5}$
5. $\frac{2}{x} + \frac{3}{x}$
6. $\frac{5}{x} + \frac{3}{2x}$
7. $\frac{3}{2x} + \frac{5}{3x}$
8. $\frac{7}{3y} + \frac{1}{y}$
9. $\frac{1}{x} - \frac{1}{2x}$
10. $\frac{b}{10} - \frac{2b}{35}$
11. $\frac{x}{18} - \frac{1}{27}$
12. $\frac{x}{5} - \frac{1}{4}$
13. $\frac{3}{x} + \frac{8}{2x}$
14. $\frac{2m}{8} + \frac{4m}{12}$
15. $\frac{s}{3} + \frac{s}{5} + \frac{s}{2}$
16. $\frac{4}{3a^2} + \frac{1}{a^2}$
17. $\frac{5}{x} + \frac{3}{x^2}$
18. $a - \frac{b}{c}$
19. $\frac{5}{x} + \frac{3}{2x} - \frac{4}{3x}$
20. $\frac{4}{a} - \frac{3b}{ab}$

Example:

$$\frac{1}{x} + \frac{1}{y}$$

The L.C.M. of x and y is xy,

$$\therefore \frac{1}{x} = \frac{1 \times y}{xy} = \frac{y}{xy}$$

$$\text{and } \frac{1}{y} = \frac{1 \times x}{xy} = \frac{x}{xy}$$

$$\therefore \frac{1}{x} + \frac{1}{y} = \frac{y + x}{xy}$$

EXERCISE 4J

Simplify the following:

1. $\frac{1}{a} - \frac{1}{b}$
2. $\frac{1}{x} + \frac{1}{2y}$
3. $\frac{3}{2m} - \frac{2}{p}$
4. $\frac{5}{2a} + \frac{1}{3b}$
5. $\frac{2v}{14} + \frac{3w}{28}$
6. $\frac{5a}{5} - \frac{2b}{25}$
7. $\frac{3e}{2} + \frac{2ef}{8}$
8. $\frac{5a}{2a} - \frac{2}{a}$
9. $\frac{a}{b} - \frac{c}{d}$
10. $\frac{x}{y} + 1 + \frac{a}{b}$

Example:

A boy buys n oz of sweets, of which x oz are chocolates, and the rest is toffee. What fraction of the total is toffee?

This is one of those questions which become very easy to solve if you invent a similar, very easy one with numbers. For example, a boy buys 8 oz of sweets, of which 3 oz are chocolate. What fraction of the total is toffee? Now it is easy:

If 3 oz are chocolate, 5, or 8 — 3, are toffee.

5 out of a total of 8 oz are toffee; in other words $\frac{5}{8}$.

Let us do the example in the same way:

If x oz are chocolate $n - x$ oz are toffee.

$n - x$ oz out of a total of n oz are toffee, in other words $\frac{n - x}{n}$

This method of trying easy numbers is always very useful when you are in doubt.

EXERCISE 4K

1. What is $2\frac{1}{2}$ inches in feet? $2\frac{1}{2}n$ inches in ft? ($2\frac{1}{2}n$ can also be written $\frac{5n}{2}$).

2. What is $12\frac{1}{2}$p in £? $12\frac{1}{2}x$p in £?

3. If there are p pupils at a school, and n are over 14 years old, how many are under 14? What fraction of the total number are over 14? Under 14?

4. A classroom is 21 ft wide and y ft long. What is its area in *square yards*?

5. $2n$ boys each had an ice cream. The total cost was 30p. How many pence did each ice cream cost?

6. 3 pencils cost 10 pence. What do n pencils cost? What do $(n + 1)$ pencils cost?

7. A reservoir contains x gallons of water. How much is left after $\frac{x}{4}$ gal is used? After $\frac{y}{z}$ gal?

8. If 5 miles equals 8 km how many km is n miles? How many is $4n$ miles?

9. A kennel owner has x puppies and sells n of them. How many remain? What fraction of the whole has she sold? What fraction are unsold?

10. What fraction of a journey remains when $\frac{x}{y}$ of it has been done?

11. Three partners share the profit in their company. The first has $\frac{2}{5}$, the second $\frac{3}{7}$, what fraction does the third get? If the first had:

$$\frac{x}{2y}, \text{ and the second } \frac{x}{3y},$$

what fraction would the third have?

12. A woman sorts $3n$ eggs every hour. How many hours would she take to sort $15x$ eggs? How many minutes?

13. A carpet is $\frac{2n}{3}$ ft long and $\frac{n}{5}$ ft wide. Find its area in ft^2 and its perimeter in ft.

14. A tea chest is a in high, $\frac{2a}{3}$ in wide, and $\frac{3a}{4}$ in long. Find its volume, in in^3. Can you find the total area, in square inches, of all six surfaces?

15. A swimming pool is $\frac{5n}{2}$ m long, $\frac{2n}{3}$ m wide, and of the same depth throughout. It contains $\frac{10n^3}{6}$ m^3 of water. What is the depth of the water in metres?

CHAPTER 5

FORMULAE AND PROBLEMS

WE HAVE already met (Chapter 2) the formula $R = \frac{1}{2}(24 - T)$ for stairs, where R is the height of the riser, and T is the length of the tread. All we have had to do so far is to put in the given length of the tread, and by working out the formula, find the riser height. But suppose we are told the height of the riser, and asked to find the tread? We need to change

$$R = \tfrac{1}{2}(24 - T)$$

into another formula which begins with T. We have to twist the old formula into a new shape. Now, of course, formulae are equations, and this can easily be done:

$$\text{If } R = \tfrac{1}{2}(24 - T)$$
$$\text{Then } 2R = 24 - T$$

(If *half* a number $= R$, the *whole* number $= 2R$)

$$\therefore T = 24 - 2R,$$

and T is now the subject of the formula.

The method used is exactly the same as that for solving equations. We can add, subtract, multiply or divide, provided we do it to both sides of the formula.

EXERCISE 5A

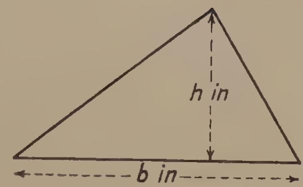
1. If W is the weight in pounds and P is the number of pints then, remembering the rhyme "*A pint of water weighs a pound and a quarter,*" $W = \frac{5P}{4}$. Rewrite the formula with P as the subject.

2. The formula for the sum of the interior angles of a polygon is $r = 2n - 4$, where r is the number of right-angles, and n is the number of sides. Rewrite the formula with n as the subject. If the sum of the angles is 4 right-angles, how many sides has the polygon?

3. If $a°$, $b°$, $c°$, are the angles of a triangle, then $a + b + c = 180$. How big is $\angle a$?

4. If the area (A cm²) of a rectangle of length L cm and breadth B cm is given by $A = LB$, what is L if $A = 42$, $B = 4$?

5. If the perimeter of the same rectangle is P cm, $P = 2(L + B)$. Make B the subject of this formula.



6. The area of this triangle is A in^2. $A = \frac{1}{2}(bh)$. What is the height in terms of A and b?

7. $V = \frac{15x}{22}$ is the formula for converting ft/s to mile/h
where V = speed in mile/h
and x = speed in ft/s.
Rewrite the formula with x as the subject, and convert 45 mile/h into ft/s.

8. The amount of simple interest is given by the formula $I = \frac{PRT}{100}$. Where £I is the interest on £P for T years at R% per annum, make R the subject of this formula and find what rate of interest is necessary to pay £16 interest on £50 if it is invested for 8 years.

9. The circumference of a circle is given by the formula $C = 2\pi r$, where C is the circumference in inches, and r is the radius in inches. Make r the subject of this formula.

10. Make y the subject of the formula $x = y(1 + a)$.

Isolating the Unknown Quantity

Sometimes the unknown is not so simple to isolate. Suppose we have an equation like this:

$$\frac{1}{7} + \frac{1}{9} = \frac{1}{x}$$

and we want to rearrange it in terms of x. We must first clear the equation of fractions. The L.C.M. of 7, 9 and x is $63x$.

$$\text{If } \frac{1}{7} + \frac{1}{9} = \frac{1}{x}$$

$$\text{Then } 9x + 7x = 63$$

$$\therefore 16x = 63$$

$$\therefore x = \frac{63}{16}$$

It is clear that solving an equation is the same thing as rearranging it in terms of x, and we have isolated x on one side of the equation.

Now one of the standard optical formulae is $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$, where u inches is the distance from the lens of the object photographed, v inches is the distance of the film from the lens, and f inches is the focal length of the lens. Suppose

we want f to be the subject of the formula, in other words we wish to isolate f on one side of the formula. Let us follow the same pattern:

$$\text{If } \frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

Then $vf + uf = uv$ (we have multiplied each term by uvf)

$\therefore f(v + u) = uv$, since f is common to both vf and uf.

Dividing both sides by $(v + u)$,

$$f = \frac{uv}{v + u}$$

Example:

$d = \sqrt{\frac{(3h)}{2}}$ is the formula for the distance (d miles) to the horizon from a height of h ft. Make h the subject of the formula.

Now $d = \sqrt{d^2}$. So we could write $\sqrt{d^2} = \sqrt{\frac{(3h)}{2}}$.

It is obvious that if the square root of one number is equal to the square root of another, then the numbers themselves must be equal. So we can write

$$d^2 = \frac{3h}{2}$$

$$\therefore 2d^2 = 3h \therefore \frac{2d^2}{3} = h$$

Fxample:

$$\text{Make } G \text{ the subject of } t = 2\pi\sqrt{\left(\frac{L}{G}\right)}$$

$$\text{Dividing by } 2\pi \therefore \frac{t}{2\pi} = \sqrt{\left(\frac{L}{G}\right)}$$

$$\text{Squaring both sides} \therefore \left(\frac{t}{2\pi}\right)^2 = \frac{L}{G}$$

$$\therefore \frac{t^2}{4\pi^2} = \frac{L}{G}$$

Multiplying by $4\pi^2G$ (the L.C.M.) $\therefore t^2G = 4\pi^2L$

$$\text{Dividing by } t^2 \therefore G = \frac{4\pi^2L}{t^2}$$

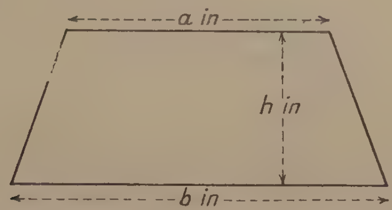
EXERCISE 5B

1. The area of a circle is given by the formula $A = \pi r^2$, where $A =$ the area in square inches, r the radius in inches. Rewrite the formula in terms of r.

2. $F = 32 + \frac{9c}{5}$ is the formula for converting temperature in degrees Centigrade to degrees Fahrenheit. Write the formula for converting Fahrenheit to Centigrade.

3. $n = \dfrac{8W}{G}$ gives the approximate life of an iron bar in sea water when $n =$ life in years, $W =$ the weight in lb per ft length and $G =$ the girth of the bar in feet. If a bar weighs 10 lb/ft and its life is 120 years, what was its original girth?

4.

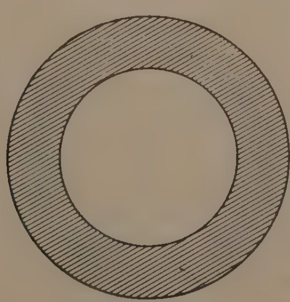


A is the area in square inches of a trapezium where $A = \frac{1}{2}h(a + b)$. Rewrite the formula in terms of a, the shorter side, and find the length of the shorter side of a trapezium of area 20 in^2, height 4 in, and whose longer side (b) is 7 in in length.

5. Make x the subject of the formula $(x - 1)^2 = t$.

6. The load that can be supported in safety by a certain kind of steel cable is given by the formula $L = 8d^2$ where L is the load in tons, and d the diameter in inches. Rewrite the formula in terms of d and find the diameter of a cable strong enough to support a load of 32 tons in safety.

7.

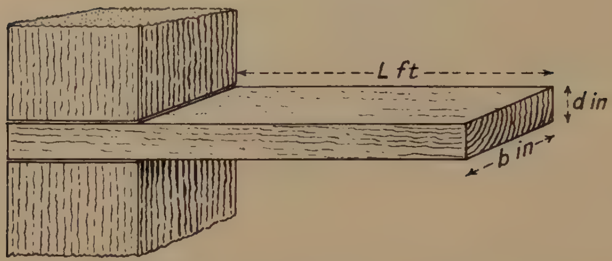


The shaded area between these two circles is called an annulus, and its area is given by the formula $A = \pi(R^2 - r^2)$, where $A =$ the area of the annulus in square feet, R the radius in feet of the larger circle, and r the radius in feet of the smaller circle. Rewrite the formula in terms of R. A circular lawn of area 628 ft^2 surrounds a circular pool of radius 5 ft. What is the radius of the lawn? (Let $\pi = 3{\cdot}14$.)

8. Make f the subject of the formula $s = ut + \frac{1}{2}ft^2$.

9. Make t the subject of the formula $s = \dfrac{t(u + v)}{2}$.

10.



An oak beam can stand a weight of W cwt before breaking, and $W = \frac{5bd^2}{4L}$, where b = the breadth in inches, d = the thickness in inches and L = the length in feet. Rewrite the formula in terms of

(i) b, (ii) d, (iii) L.

The Purpose of Equations

Mathematicians are constantly being asked to solve problems. Designing telescopes, building skyscrapers, circuits for radio and computers, the way things grow, all involve calculations, some of them very complex. Almost all of these calculations use equations, and the ability to solve and manipulate equations is one of the mathematician's and engineer's most useful tools. Indeed sometimes by calculation the mathematician has been able to point to the existence of something previously unknown.

When you turn on the radio, remember that Maxwell in 1864 constructed an equation which led him to believe that there were electrical or radio waves, which spread like sound waves. It was another twenty years before another scientist actually made any, and proved that Maxwell's theory was right. Maxwell's equation had told him that radio waves *ought* to exist.

How can we use equations to solve problems? Let us start with a simple one.

Example:

Bob and John collect conkers. Bob has three times as many as John, but it works out that if he gives John six of his, he will then still have twice as many. How many did each have at first?

First of all, we must ask ourselves, "What are we asked to find?" If we give this unknown quantity a name, x, or a, or any letter we like, and then attempt to make an equation from the information in the question, we ought to be able to find the unknown by solving that equation. In this case we have to find two answers, the number of conkers that each has, but since one depends on the other, i.e., Bob has three times as many as John, if we find one, we know the other.

So let the number owned by Bob be x conkers.

Then John, who has only $\frac{1}{3}$ the amount, must have $\frac{x}{3}$.

Bob gives 6 to John, and has $\therefore x - 6$ left

and John now has $\frac{x}{3} + 6$,

Bob still has twice as many, so $x - 6$ is twice $\frac{x}{3} + 6$.

Now can we make an equation from this information?

If we divide $(x - 6)$ by 2, it will be *equal* to $\frac{x}{3} + 6$.

$$\therefore \frac{x-6}{2} = \frac{x}{3} + 6$$

Solving the equation, $3(x - 6) = 2x + 36$

$$\therefore 3x - 18 = 2x + 36$$

$$\therefore x = 54$$

Now this is not the answer to the problem, but only the solution of the equation. x stood for the number of conkers owned by Bob. Since John had $\frac{1}{3}$ the number he must have 18.

Bob has 54.
John has 18.

Check: If Bob has 54 and gives 6 to John he is left with 48 and John has 24. Bob therefore *does* have twice as many.

These are the steps you should take to solve a problem:

1. Read the question carefully, asking yourself what is the unknown you are asked to find. It is often useful to draw a diagram.
2. Choose a letter to represent the unknown number, but, remember that it stands for a number and not a quantity. Don't write "Let *a* be the length," but "Let the length be *a* in. (or ft., or miles, whichever is suggested by the question). Don't write, "Let x be the speed," but "Let x mile/h be the speed." Since there are some problems with mixed units, e.g. feet *and* inches, it is as well to be clear about which you are using.
3. Write down all the facts of the problem using the unknown you have chosen, and assemble them in the form of an equation.
4. Solve the equation.
5. Remember that the solution of the equation is not necessarily the answer to the problem. Write down the answer clearly.
6. Check the original question, *not* in your equation which may be wrong. If you have correctly solved the wrong equation it does not help you to solve the problem.

Let us see how this works with a harder problem:

Example:

Mr. Bacon drove from Cambridge to York. He left at 8 a.m., drove at an average speed of 30 mile/h, which did not include the half-hour when he stopped for coffee. His nephew, Augustus Egg, who drove a sports car, left at 9 a.m., and took a country route which was 10 miles longer. He averaged 40 mile/h, did not stop for coffee, and arrived half an hour before his uncle. How far was it from Cambridge to York by the direct route?

This is one of an important class of problems involving speed, time and distance. It is useful to remind ourselves about the relationship between them. If in doubt, use very simple figures, and notice what is done. For example:

If the *speed* is 30 mile/h and the *distance* 60 miles, the *time* is $\frac{60}{30} = 2$ hours

If the *time* is 2 hours and the *distance* 60 miles, the *speed* is $\frac{60}{2} = 30$ mile/h

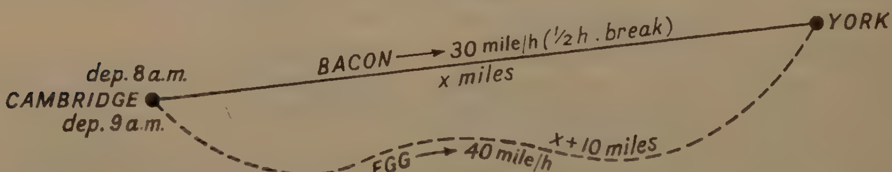
If the *speed* is 30 mile/h and the *time* is 2 hours, the *distance* is

$30 \times 2 = 60$ miles

In other words, if $t =$ time, $d =$ distance, $v =$ speed,

$$\text{then } t = \frac{d}{v},\ d = vt,\ v = \frac{d}{t}$$

To return to our problem. Draw a diagram:



In this problem the question asked is "How far was it from Cambridge to York?" The speeds are in miles per hour, the times in hours, so it seems sensible to use miles as the unit of distance.

Let the distance by the direct route be x miles.

Bacon's speed is 30 mile/h

$\therefore$ his travelling *time* is $\frac{x}{30}$ hours

$\therefore$ his total time, since he stopped for ½ hour, is $\left(\frac{x}{30} + \frac{1}{2}\right)$ hours

Egg's *speed* is 40 mile/h.

The *distance* he covers is $(x + 10)$ miles, since he went by a longer route.

$\therefore$ his travelling *time* is $\frac{x + 10}{40}$ hours.

But he took $1\frac{1}{2}$ hours less than Bacon, since he started an hour later and arrived $\frac{1}{2}$ hour earlier.

In order to make an equation, therefore, or to make his time *equal* to Bacon's, we must *add* $1\frac{1}{2}$ hours to his time:

$$\frac{x+10}{40} + 1\tfrac{1}{2} = \frac{x}{30} + \frac{1}{2}$$

Solving the equation, $3(x+10) + 180 = 4x + 60$

$$\therefore\ 3x + 30 + 180 = 4x + 60$$
$$\therefore\ 150 = x$$

The distance from Cambridge to York is 150 miles.

Check: If the distance was 150 miles,
Bacon took 5 hours + $\frac{1}{2}$ hour, and arrived at 1.30 p.m.
Egg took 4 hours, and arrived at 1 p.m., i.e. half an hour earlier.

EXERCISE 5C

This exercise is the best test of your understanding so far. Work every question and if you are stuck, get help; don't give up.

1. Jim was new to the farm, and looking round the yard saw that there were cows and turkeys and no other animals. He noticed that there were 19 more turkeys than there were cows and that there were 92 legs altogether, not counting his own. "Ha!" he said (having been very good at mathematics when at school). "Now I can tell how many of each there are." Can you?

2. If the length of a rectangle is 32 m more than the width, and the perimeter is 424 m, find the length and the width.

3. Arthur was not a good shot. His friends promised him 10p every time he hit the target, but asked him for 5p every time he missed. He had six more misses than hits and made 90p. How many hits did he make?

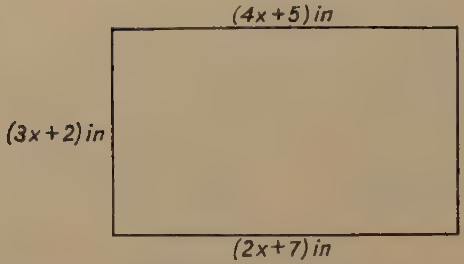
4. If you take 6 from a number and divide the result by 5, you will get the same answer as taking 4 from it and dividing the result by 7. What is the number?

5. Two brothers were cycling. The elder rode at an average speed of 13 mile/h, the younger at 9 mile/h. How long was it before the elder was 10 miles ahead?

6. A bricklayer worked for 5 days building a wall. His mate, who was only paid $\frac{3}{5}$ as much as the bricklayer, worked for only 3 days. If they earned £10·20 between them, what were their daily wages?

7. The rule in some mathematics classes is 2 marks for a sum right, and (-1) for a sum wrong. If a boy does 23 sums and scores 13 marks, how many sums did he get right?

8.



In this rectangle, find (i) x; (ii) the length of the perimeter.

9. If $\frac{3}{8}$ of a number plus $\frac{1}{8}$ of it is equal to 15 minus the number, what is it? If $\frac{3}{8}$ of the number plus $\frac{1}{8}$ of it is *less* than 15 minus the number, can you say anything about it?

10. A confectioner makes a mixture of 30 lb. of chocolates. Those with hard centres sell at 40p per lb, those with soft centres at 50p per lb. How many pounds of each kind must be mixed if the mixture is to sell for 48p per lb?

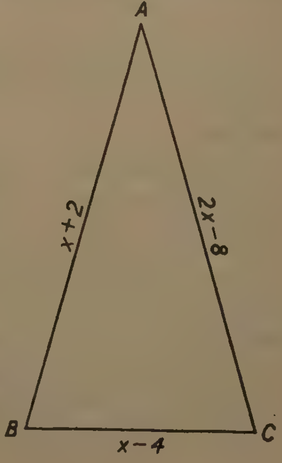
11. What number added to both the numerator and denominator of the fraction $\frac{9}{22}$ changes it to a fraction equal to a half?

12. A new car was being tested on an autobahn. The car travelled from Cologne to Frankfurt at an average speed of 128 km/h, but on the return journey it averaged only 96 km/h because of bad weather. If the whole journey took $3\frac{1}{2}$ hours, how many kilometres is it from Cologne to Frankfurt?

13. Find four consecutive odd numbers whose sum is 64.

14. A jet plane travels 11 times as fast as an express train. In 1 hour the jet aircraft covers 150 miles more than the train does in 8 hours. How fast is the plane? The train?

15. In $\triangle ABC$, $AB = AC$. The lengths of the sides are in inches. What is the perimeter of the triangle?

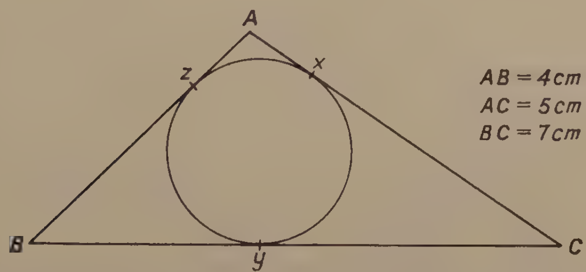


16. Antrobus was a cyclist, and set off from Barnstaple to Bampton, 28 miles away. His average speed was 12 mile/h. His father, who liked to walk, set off at the same time from Bampton towards Barnstaple, and he averaged 4 mile/h. After how many hours did they meet?

17. A savings box contains 47p, made up of 1p, 5p, and 10p coins. If the number of 5p coins is twice the number of 10p coins and there are

3 more 1p coins than 5p coins, how many of each coin are there in the box?

18.



In the $\triangle\, ABC$, the inscribed circle touches the sides at X, Y and Z. If $AX = AZ$ (see Book 2, Chapter 2, tangents to a circle from an external point), $BZ = BY$, and $CX = CY$, find the length of AX.

19. There were 156 people employed in a television factory. The men got £3·50 a day, the woman £2·50 a day. The accountant found that his daily wage bill came to exactly £440. How many of each were there?

20. Roberts lives 1 mile away from his school. He has to be at school by 9.15. If he leaves home at 3 minutes past nine, walks at 3 mile/h, and runs at 7 mile/h, how many minutes can he afford to walk if he is not to be late?

EXERCISE 5D

1. A greengrocer, looking at a consignment of 7 dozen oranges, found that some were better than others. The better ones he sold in lots at 15 for 35p, the others at 3 for 5p. Altogether he sold them for £1·80. How many of each kind were there?

2. In the formula $$s = \frac{t\,(u + v)}{2},$$
the value of s when $t = 3$, $v = 4$, is one less than when $t = 2$, $v = -2$. Find the value of u.

3. The cost of a journey from London to Paris by air is half as much again as the cost by train and boat. If the total cost of 12 tickets by air and 8 by train is £208, find the cost of an air ticket.

4. Jane took two tests at the end of term. In the History test she only lost 4 marks in a paper with 6 questions. In the Geography test she lost 7 marks in a paper with 9 questions. Her average History mark was one more than her average Geography mark. Both papers had the same maximum possible marks. Can you say what it was?

5. Mr. Fish and his old friend Alderman Chips saved up for a holiday together. Before they left they discovered that Mr. Fish had £5 more than his friend. In addition the Alderman lost a £5 note on the train. After a week at Blackpool they counted their money. Mr. Fish found he had spent $\frac{8}{11}$ of his, and Alderman Chips had spent $\frac{4}{5}$ of what he arrived with. They also discovered that Mr. Fish had £6 more than the Alderman. How much did each start with?

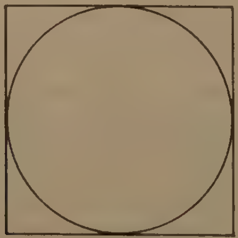
6. When doing a calculation, I found that by mistake I had taken away 7 from a number and divided by 9, instead of taking away 9 and dividing by 7. When I checked I found that my wrong calculation gave me an answer 2 less than the correct result. What was the number?

7. At 9.28 a.m. a goods train leaves London northbound, and averages 25 mile/h. At 10 a.m. the passenger express leaves on the same route, averaging 75 mile/h. At what time will the express overtake the goods train?

8. Two planes were entered in an air race. One was half as fast again as the other. The faster received a handicap of 35 minutes and caught up with the slower after 420 miles. What is the average speed of the slower plane?

9. Franz said to his old friend Hans, "Four years ago you were four times as old as I, and now you are only three times as old. Soon we will both be the same age." "I am afraid not," said Hans. "But from what you have told me, I know your present age." Can you say how old is Franz?

10.



A circle is inscribed in a square. The two perimeters total 21·42 in. If $\pi = 3{\cdot}14$ approximately, what is the diameter of the circle?

CHAPTER 6

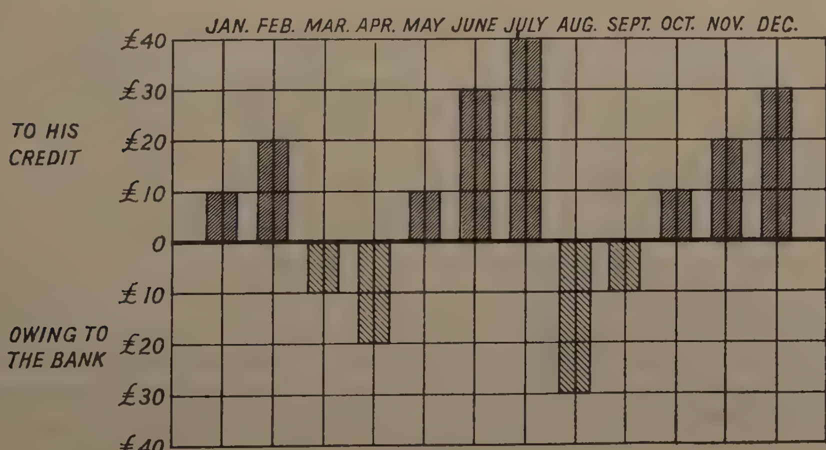
DIRECTED NUMBERS

YOU HAVE probably read that in certain parts of the world, in Canada, for example, or Siberia, the winter temperatures are as low as $-30°$ Centigrade, and you have assumed, quite rightly, that this means 30° below zero, or freezing point.

Now this way of using the minus sign is quite different from the way in which we have used it so far. In this sum: $108 - 54$, the minus sign is an *instruction*. It tells us that 54 is to be taken away from 108. But $-30°$ means a temperature 30° C. *colder* than zero (just as 20° C. means 20° C. *warmer* than zero), and is a number less than 0.

This is perhaps a strange idea at first. How can there be numbers less than 0?

It is quite true of course, that for some things there *are* no numbers smaller. A man can have one pet, or no pet, he can't have less than none. A room can have nothing in it, but can't have less than nothing. But this does not apply to everything. With fractions, it would be absurd to talk about $1\frac{2}{7}$ of an elephant, but quite sensible to think of $7\frac{1}{2}$ cwt of coal. In the same way the idea of numbers less than 0 is sensible when we are talking about *movement*. Suppose we make a diagram of the amount of money a man has in his bank during one year:



In January he has £10 in the bank.

In February he pays in £10, and now has £20.

In March he draws out £30. Now, not only has he no money left, but owes the bank £10.

In April he draws out £10, and now owes the bank £20.

In May he puts in £30, and now has £10.

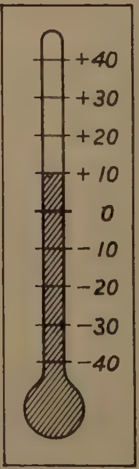
In June he puts in another £20, and now has £30 in the bank, and so on.

You can see from the diagram that sometimes he has a balance, but sometimes he owes the bank money. In August he not only has no money in the bank, but owes them £30. We could say that he is — £30 well off. Here is a case where numbers less than 0 have a real meaning, and a minus or plus sign can be used to indicate direction, up or down, from a given point, often, and in this case, 0.

So + £20 means he has a balance of £20, and — £20 means he has an overdraft of £20.

The most common use of the + and — signs in this way is in the Centigrade thermometer where 0° is the melting point of ice.

The temperatures above zero are marked + 10°, + 20°, + 30°, and those below zero, — 10°, — 20°. Because the signs show the *direction* (up or down), these are called DIRECTED NUMBERS. 0 means there is no movement either way, so does not need either + or —.



+ 10 is a positive number,
— 10 is a negative or minus number.

Now suppose the temperature is + 10° C., and falls 20° (that is, 20° is taken off). What is the new temperature? By looking at the diagram it is easy to see that it must be — 10°.

We can write this + 10 — + 20 = — 10.

But this looks rather odd, so there are a number of ways in which we can show the difference between signs used as *verbs*, add or subtract, or as adjectives, positive or negative.

Here are two of them:

We can write $^{+}10 - {}^{+}20 = {}^{-}10$, raising the directing sign,
or, (+ 10) — (+ 20) = (— 10), using brackets.

If the temperature falls another 15°, it will then be — 25° C. Notice that — 25° C. is colder than — 10° C.

For many hundreds of years mathematicians came across sums which needed minus numbers, and considered that the answers were impossible or absurd, but Leonardo da Pisa, who lived in the thirteenth century,

explained a negative answer by saying that it represents a debt, and by the seventeenth century the idea had been accepted. It was found that negative numbers were very much like ordinary numbers and could be added, subtracted, multiplied or divided.

Directed numbers can be used to show gain or loss of time, or height, or weight, or other movements, up or down, or backwards and forwards.

Example:

If I start from a point and walk 4 miles northwards, this could be represented as + 4 miles. If I then turn round and walk 2 miles southwards, we could either describe this as (+ 2 miles) southwards, or, *using the northward direction as the positive one*, as (− 2) miles northwards. In the same way, a high diver's descent can be traced from 20 ft, to 15, 10, 5 and 0 ft, above water level and then as he descends to the bottom of the bath, top (− 7) ft *above* water level (or of course, (+ 7) ft *below* water level).

EXERCISE 6A

1. By using the Centigrade thermometer shown on page 198 say what is meant by (+ 32)°, (− 15)°? How would you write, with directed numbers,

(a) 20° above zero?

(b) 35° below freezing point?

(c) $2\frac{1}{4}$° below freezing point?

2. If a man has £70 in the bank, and this can be written as £(+ 70) what will be the balance if he draws out £20? If he then draws another £60, how much will his *balance* be? (Not his overdraft.)

3. Explain the meaning of this, using numbers without signs:

(a) Some parts of Holland are (− 5) m above sea-level.

(b) By going on a diet, I have gained (− 5) lb in weight.

(c) My brother is (− 2) years older than I.

(d) That watch is (− 12) minutes fast.

(e) The First World War started (− 25) years after the Second World War.

(f) His house is (− 3) miles north of mine.

4. A motor-car dealer gains or loses the following amounts when selling motor-cars. Use directed numbers to show what *profit* he makes in each case.

Austin, profit £50.	Rolls Royce, loss £56.
Ford, profit £10.	Lagonda, loss £185.
Hillman, loss £25.	Morris, profit £30.

5. A football match was due to start at 2.30 p.m. If the following numbers represent minutes after the advertised time, say what time the match actually started in each instance:

− 25, − 12, − 2, 0, + 20.

6. The average weight of some children in a class was 8 stone. By using directed numbers, show by how many pounds each of the following was *above* the average:

John	7 stone 12 lb.
Peter	8 stone 6 lb.
Jane	9 stone 1 lb.
Betty	7 stone 2 lb.
Simon	8 stone.
Mark	7 stone.

7. In a shooting contest James reckons he can score 15 out of 20 in each round. His first four scores are 16, 13, 15, 12. Using directed numbers, say how much he was above his own estimate for each round. How many must he score in the fifth and final round to bring his average score up to 15?

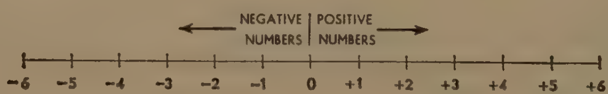
8. The following Centigrade temperatures were recorded each hour during one day in a cold greenhouse:

TIME	12 noon	1	2	3	4	5	6	7	8	9	10	11	12 mid-night	1	2	3	4	5	6	7	8	9	10	11
TEMP. °C	+10	+11	+12	+10	+10	+7	+5	+2	+2	+1	+1	0	0	−1	−3	−5	−5	−5	−4	−3	−1	0	+3	+5

Draw a graph to show these temperatures, and say at what times the owner should have turned the heat on and off if his plants were not to have been killed by frost.

Addition and Subtraction

We said earlier that directed numbers can be added and subtracted in much the same way as ordinary numbers. A simple way to see how this is done is to use a number scale:



It is easy to see what happens if we do a simple addition:

$4 + 2$, we move to the right.

Put your finger on $(+4)$, count two more to the right and we arrive at $(+6)$.

For subtractions, we move to the left, $4 - 2 = 2$.

Now what happens when we want the answer to $(+4) + (-2)$? Start from $(+4)$ on the scale, but move *backwards* two steps, in the negative number direction. We arrive at $(+2)$.

$$\text{So } (+4) + (-2) = +2.$$

It is rather as if you had £4 on your desk and the postman suddenly *added*

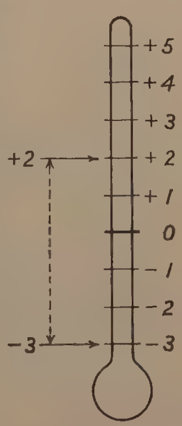
a bill for £2. You are then only really worth £2. $(+4) + (-2)$ is the same as $4 - 2$. *Adding a minus number is the same as subtracting a positive number.* Suppose you only had £1, and the postman added a bill for £3, in other words £$(+1)$ + £(-3). How much would you be worth? Starting from $(+1)$ on the number scale and counting backwards you would find that you are worth £(-2), or are in debt for £2. But supposing you owed £2 already, and received a bill for £4, in other words £(-2) + £(-4). Counting back on the scale, you are worth £(-6), or your debt would be £6 altogether.

What happens if we subtract? Consider the following sum:

$$(+2) - (-3)$$

This is equivalent to saying, "I have £2 and I have just learnt that a debt which I had for £3 is not mine after all. So *I am* £3 *better off than I thought*. In other words $(+2) - (-3) = (+5)$.

What we are asking here is, "What is the difference between $(+2)$ and (-3)?" Let us suppose they are temperatures, draw a thermometer and mark in the two quantities: Just as in the sum $7 - 3 = 4$ for example, we are saying, "To get from 3 to 7 we have to add 4, so it can be seen from the diagram that to get from -3 to $+2$ we have to add 5.



On the number scale, start at $(+2)$, turn left to subtract, and go *backwards* (-3 is a negative number) for 3 units. We will arrive at $+5$.

In the same way $(+2) - (+3)$ will mean, start at $(+2)$, turn left (subtracting) and go *forwards* ($+3$ is positive) 3 units, which brings us to (-1).

$$\text{And } (-2) - (-3) = +1$$

would be like saying that a man who owes £2 is £1 better off than a man who owes £3.

What do the following statements mean? Check with the number scale.

1. $(+3) + (+4) = (+7)$.
2. $(+2) + (-1) = (+1)$.
3. $(-3) + (+1) = (-2)$.
4. $(-5) + (-4) = (-9)$.
5. $(-5) - (+4) = (-9)$.
6. $(+5) - (-2) = (+7)$.
7. $(-6) - (-4) = (-2)$.
8. $0 + (+2) = (+2)$.
9. $0 - (+2) = (-2)$.
10. $-(-5) = (+5)$.

We can make rules to simplify this, and so that we do not have to puzzle it out each time. They are known as the *Rule of Signs* (n stands for *any* number).

(i) $+(+n)$ is the same as $(+n)$, or move to the right on the scale.
(ii) $+(-n)$ is the same as $(-n)$, or move to the left on the scale.
(iii) $-(+n)$ is the same as $(-n)$, or move to the left on the scale.
(iv) $-(-n)$ is the same as $(+n)$, or move to the right on the scale.

You will notice in (i) and (iv), where there are two similar signs, the result is positive, and in (ii) and (iii), where the signs are different the result is negative. Knowing these rules gives us the opportunity to drop the brackets very often, where it can be done without changing the meaning.

EXERCISE 6B

Find the value of these expressions:

1. $(+5) + (+7)$.

2. $(+8) + (-4)$.

3. $(+3) - (+4)$.

4. $(+12) - (-2{\cdot}5)$.

5. $(-4) + (+6)$.

6. $(-17) + (+4)$.

7. $(-2) + (-3)$.

8: $(-5) + (-1)$.

9. $(-13) - (+4)$.

10. $(-6) - (-3)$.

11. $(-2) - (-4) - (-6)$.

12. $(-4) + (-4)$.

13. What must you add to $(+3)$ to give $(+2)$?

14. What must you add to (-4) to give (-6)?

15. $0 - (+3)$.

16. $0 + (-4)$.

17. If $x = 2, y = -3$, find the value of:
(i) $x + y$,
(ii) $x - y$,
(iii) $2x + 2y$,
(iv) $x - 2y$.

18. Simplify:
(i) $a - 2a$,
(ii) $2a - 7a$,
(iii) $-a - 3a$,
(iv) $a - (-a)$.

19. Simplify:
(i) $x + 3y - 2x - 2y$,
(ii) $2x + y - 2x - 4y$,
(iii) $2a - b - a - 2b - a$.
(iv) $(+4a) - (+2a)$,
(v) $(-3a) - (-a)$.

20. Simplify:
(i) $(+b) - (-a)$,
(ii) $(+7y) - (+9y)$,
(iii) $(-x) - (+x)$,
(iv) $0 - (-2n)$,
(v) $(+a) - (+5a)$.

Multiplication and Division

If £2 × 3 means we have three amounts of £2, or £6 altogether, then $(-2) \times 3$ must mean that we have three *debts* of £2, or a *balance* of £(-6).

So $(-2) \times 3 = (-6)$,

$2 \times (-3)$ must mean the same thing, two debts of £3,

so $2 \times (-3)$ also equals (-6).

What about $(-2) \times (-3)$? This is equivalent to saying that a debt for £2 has been *taken away* three times, in other words, we are £6 better off than we would have been if we had not had the debts taken. In other words we are £6 to the good or £$(+6)$. So it looks as though $(-2) \times (-3) = (+6)$.

Example:

Let's see if we can do this algebraically. It is not difficult to check for yourself that $(x - 1) \times (x - 2) = x^2 - 3x + 2$, by taking, for instance $x = 4$.

$$\begin{aligned} \text{If } x = 4, \text{ then } x - 1 &= 3 \\ \text{and } x - 2 &= 2 \\ \therefore (x - 1) \times (x - 2) &= 6 \\ \text{and } x^2 - 3x + 2 &= 16 - 12 + 2 \\ &= 6. \end{aligned}$$

Try it yourself with 5, 3, 2, 1.

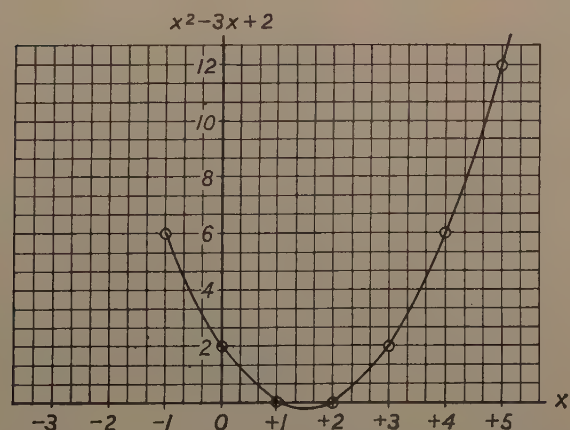
Now what happens if $x = 0$?

$$\begin{aligned} \text{Then } x^2 - 3x + 2 &= 0 - 0 + 2 \\ &= +2 \\ (x - 1) &= 0 - 1 \\ &= -1 \\ \text{and } x - 2 &= 0 - 2 \\ &= -2 \end{aligned}$$

If $(x - 1) \times (x - 2)$ is to equal $x^2 - 3x + 2$, it looks as though $(-1) \times (-2)$ must also equal $(+2)$.

We could plot the values of x in the expression $x^2 - 3x + 2$ and make a graph:

When x =	5	4	3	2	1	0	−1
Then $x^2 - 3x + 2$ =	12	6	2	0	0	2	6



Can you guess from the graph what $x^2 - 3x + 2$ is equal to when $x = -2$? Do you think there are any values of x which give a negative value for $x^2 - 3x + 2$?

These illustrations seem to show that assuming the product of minus numbers to be plus numbers gives reasonable results. We can list the rules like this:

1. Two positive numbers multiplied give a positive product,

   or $(+x) \times (+y) = +xy$, usually written xy.

2. Two negative numbers multiplied give a positive product,

   or $(-x) \times (-y) = (+xy)$ or xy.

3. Numbers with unlike signs give a negative product,

   $$(-x) \times (+y) = (-xy), \text{ or } -xy$$
   and $(+x) \times (-y) = -xy$.

When directed numbers are multiplied, like signs give a plus answer; unlike signs a minus answer.

Examples:

$$(+5) \times (+4) = (+20) \text{ or } 20.$$
$$(-5) \times (-4) = (+20) \text{ or } 20.$$
$$(-5) \times (+4) = (-20) \text{ or } -20.$$
$$(+5) \times (-4) = (-20) \text{ or } -20.$$

EXERCISE 6C

Find the value of these expressions. (Notice that in No. 15 $(-x)(+2x)$ means $(-x) \times (+2x)$. The same applies to Nos. 16-20.)

1. $(-4) \times (+3)$.
2. $(-3) \times (+4)$.
3. $(-7) \times (-9)$.
4. $(-8) \times (+3)$.
5. $(+3) \times (-17)$.
6. $(-3) \times (-3) \times (-3)$.
7. $12 \times (-5)$.
8. $3 \times (-6)$.
9. $16 \times (-2)$.
10. $(+2)^2$.
11. $(-4)^2$.
12. $(-3)^2$.
13. $(-2)^3$.
14. $(-\frac{1}{2})^2$.
15. $(-x)(+2x)$.
16. $(-3x)(-x)$.
17. $(-x)(+x)$.
18. $(-xy)(+yz)$.
19. $(-3x)(-6x)$.
20. $(-3)(-4ab)$.

If $x = 3$, $y = -2$, $z = -4$, find the value of the following expressions:

21. xyz.
22. x^2y.
23. xy^2.
24. $xy - z$.
25. $x - yz$.
26. $2x - yz$.
27. $3xy - z$.
28. $xy^2 + z^2$.
29. $y^2 - z^2$.
30. $x^2 - y^2$.

If $(+4) \times (+2) = (+8)$, it follows that $(+8) \div (+4) = (+2)$.
In the same way,

If $(-4) \times (-2) = (+8)$, then $(+8) \div (-4) = (-2)$.

If $(+4) \times (-2) = (-8)$, then $(-8) \div (+4) = (-2)$.

If $(-4) \times (+2) = (-8)$, then $(-8) \div (-4) = (+2)$.

So it seems clear that the rule of signs for division is the same as that for multiplication: that is, *when directed numbers are divided, numbers with like signs give a plus answer, unlike signs a minus answer.*

You remember that $(-8) \div (-4)$ can also be written $\frac{(-8)}{(-4)}$, and here the bracket may be left off.

EXERCISE 6D

Find the value of these expressions:

1. $(-6) \div (+2)$
2. $(-18) \div (-3)$
3. $(-14) \div (+7)$
4. $(+54) \div (-6)$
5. $(-18) \div (-6)$
6. $\frac{-27}{+9}$
7. $\frac{-72}{-12}$
8. $\frac{+28}{-7}$
9. $\frac{-6}{+6}$
10. $\frac{+6}{-6}$
11. $\frac{-6}{-6}$
12. $\frac{-6}{6}$
13. $\frac{-2}{+4}$
14. $(-2n) \div (+n)$
15. $(-8x) \div (-2)$
16. $(+xy) \div (-x)$
17. $(-6x) \div (+3x)$
18. $(+6a^2) \div (-3a)$
19. $\frac{-12x^2}{-4x}$
20. $\frac{(-x) \times (-y)}{(-z)}$
21. $\frac{4x - 12}{-4}$
22. $\frac{-4x + 12}{4}$
23. $\frac{-4x - 12}{4}$

If $x = 2$, $y = -5$, $z = 0$, find the value of the following expressions:

24. $2xy$
25. $4x^2y + z$
26. $xy + xz + yz$
27. $4xy + 5yz$
28. $x^2 + y^2$
29. $\frac{x + y^2}{x}$
30. $3z^2 + 5y$
31. xy^2z
32. $\frac{xz}{y}$
33. $\frac{3x^2 + z}{y}$
34. $\frac{x}{y} \times \frac{y^2}{x^2}$
35. $\frac{z + 2x^2y}{y^2}$

Directed Numbers and Brackets

We have already learned that brackets are intended to show in what order operations are to be carried out. What happens when there are directing signs in front of a bracket? Let us look at these examples:

Example:

(i) $12 + (7 + 4)$.
This means 12 plus the sum of 7 and 4. Answer 23. Without the brackets, it would look like this: $12 + 7 + 4$, and would still equal 23. Removing the brackets has made no difference.

(ii) $12 + (7 - 4)$.
This means 12 plus the difference between 7 and 4, or $12 + 3$, which is 15. Without the brackets, $12 + 7 - 4$, this would mean, add 7 to 12 and subtract 4 from the result. Answer 15. Again, removing the brackets has made no difference.

(iii) $12 - (7 + 4)$.
This means, add 7 and 4, and take the result from 12. Answer 1. But without the brackets, it means $12 - 7 + 4$, which is 9. However, by changing the sign of the 4, like this, $12 - 7 - 4$, we once again get the right answer, 1.

(iv) $12 - (7 - 4)$.
This means, take 4 from 7, and the result from 12. Answer 9. Once again, by changing the sign of the 4, we can arrive at the correct answer without brackets. $12 - 7 + 4$ also equals 9.

(v) $12 - (7 - 3 - 1)$.
This means take 3 from 7 and 1 from the difference, and subtract the result from 12. Answer 9. Without the brackets it means $12 - 7 - 3 - 1$, or 1. However, by changing the signs within the brackets, we obtain the correct answer without brackets,

$$12 - 7 + 3 + 1, \text{ or } 9.$$

(vi) $12 - (-4 + 7)$.
This means, add 7 to (-4) and take the result from 12.

$$-4 + 7 = +3,$$
$$12 - (+3) = 9.$$

How can we achieve this result by removing the brackets? It must evidently be $12 + 4 - 7$, which equals 9, that is, by changing all the signs inside the bracket. As we have seen above, subtracting a minus number is equivalent to *adding* it, and $-(-4) = +4$.

You notice that it has been necessary to change the sign of the 4, which is *inside* the bracket, in (iii) and (iv). These are the two examples which have

minus signs outside the bracket. If you have to remove brackets, therefore, the rule is:

If the bracket has a plus sign in front of it, the signs inside remain unchanged.

If the bracket has a minus sign in front of it, ALL the signs of the terms inside are changed, $+$ *to* $-$, *and* $-$ *to* $+$.

Examples:

$$+(x+y)=x+y$$
$$+(x-y)=x-y$$
$$-(x+y)=-x-y$$
$$-(x-y)=-x+y \text{ (or } y-x\text{)}.$$

Notice:

(i) It is usual to start with a positive expression where possible.
(ii) A positive number beginning an expression usually has no sign.

Example:

Remove the brackets in this expression:

$$-(b^2+3b-4)$$

Since there is a minus sign outside the bracket, we must change *all* the signs of the numbers inside it, no matter how many there are, including that of the first term b^2, which, of course means $+b^2$.

$$\text{so } -(b^2+3b-4)$$
$$=-b^2-3b+4$$

Example:

Remove the brackets and simplify $-2(a^2+5a-6)$.

In this case we have to multiply every term in the bracket by -2:

$$(-2)\times(+a^2)=-2a^2$$
$$(-2)\times(+5a)=-10a$$
$$(-2)\times(-6)=+12$$
$$\therefore -2(a^2+5a-6)=-2a^2-10a+12$$

You may find it simpler to multiply the terms in the bracket by the number outside, and remove the bracket as a separate step, e.g.:

Step 1: $\quad -2(a^2+5a-6)=-(2a^2+10a-12)$

Step 2: $\quad =-2a^2-10a+12$

Example:

$$x+3-5(x-2)$$

In this case it is important to remember what the expression means: x plus 3, minus 5 times $(x-2)$. The contents of the bracket are multiplied by minus 5.

So $x+3-5(x-2)=x+3-5x+10$

$$=x-5x+3+10$$
$$=-4x+13 \text{ (collecting like terms), or } 13-4x$$

EXERCISE 6E

Remove the brackets and simplify:

1. $+(3-2)$
2. $-(3-2)$
3. $7+(2+4)$
4. $7-(2+4)$
5. $-3(4-6)$
6. $+(-3x)$
7. $-(-y)$
8. $-5(+2n)$
9. $-3(-7ab)$
10. $-4(x+2y)$
11. $x+(w+x)$
12. $x-(w+x)$
13. $3a+(2a-b)$
14. $2-(x+2)$
15. $3(a^2-2a+2)$
16. $-(6-5n)$
17. $n+4+3(n-1)$
18. $-(x-2)+(2x-1)$
19. $5a-(-2a)$
20. $(a+b)-(a-b)$
21. $(a+b)+(a-b)$
22. $(3x+2y)-(x+y)$
23. $(2e+5f)-(e-f)$
24. $(e-f)+(2e-5f)$
25. $(x+y)-(7x+6y)$
26. $3(a+b)-3(a+b)$
27. $5(x+2y)+2(2x-5y)$
28. $3m-(1+2m-n)$
29. $-2(x^2+3x+2)$
30. $6a-4b-5c-5(a-b-c)$
31. $2(4x+y+3z)-3(x-2y+z)$
32. $-(m+2n)-(-2m-n)+(3m+4n)$

Using Brackets

In an expression containing terms with a common factor it can be useful to reverse this process. $2x+8y$ can be written $2(x+4y)$, the factor 2 being common to both terms. Be careful when using brackets after a minus sign, and remember that the signs within the new bracket must be changed.

Example:

Complete: $a-2b-3c=a-(\ldots\ldots)$

Since there is a minus sign outside the bracket, both $-2b$ and $-3c$ must have their signs changed:

$=a-(2b+3c)$. *Note:* $2b$ is the same as $+2b$.

EXERCISE 6F

Complete the following statements:

1. $x-y=-(\ldots\ldots)$.
2. $x+2y-32=x+(\ldots\ldots)$.
3. $x-y-z=x-(\ldots\ldots)$.
4. $3x-y+6=3x-(\ldots\ldots)$.
5. $x^2-5x+6=-(\ldots\ldots)$.
6. $3m+21n=3(\ldots\ldots)$.
7. $10a-25b=5(\ldots\ldots)$.
8. $4e-2ef=2e(\ldots\ldots)$.
9. $5x^3-2x^2=x^2(\ldots\ldots)$.
10. $2a+ab=-a(\ldots\ldots)$.

Directed Numbers in Equations

You should now be able to understand that a statement like $N + 4 = 3$ makes sense if you are prepared to consider that N may be a negative number. Clearly it must be -1. Then substituting, $-1 + 4 = 3$.

Example:

I am told that the temperature has gone up 12° C. since seven o'clock. On reading the thermometer, I find it reads 8° C. What was it at seven o'clock?

Let x° C. be the temperature at seven o'clock.
Then the present temperature is $(x + 12)$° C.
But we are told that the present temperature is 8° C.

$$\therefore x + 12 = 8$$
$$\therefore x = 8 - 12$$
$$= -4° \text{ C.}$$

Example:

Solve the equation $-2(x + 3) = 12$.
Remove the bracket, $\therefore -2x - 6 = 12$

$$\therefore -2x = 12 + 6$$
$$\therefore -2x = 18$$

Divide both sides by 2, $-x = 9$

But we want to find $+x$, not $-x$. How can we change $-x$ to $+x$? By dividing it by -1. So *both* sides must be divided by -1.

$$\therefore \frac{-x}{-1} = \frac{+9}{-1}$$
$$\therefore x = -9$$

This difficulty is avoided by putting the x on the right, if that gives a positive coefficient. The sum would then look like this:

$$-2(x + 3) = 12$$
$$\therefore -2x - 6 = 12$$
$$\therefore -6 - 12 = +2x$$
$$\therefore -18 = 2x$$
$$\therefore -9 = x$$

Check: When $x = -9$

$$\text{Left-hand side} = -2(-9 + 3)$$
$$= 18 - 6$$
$$= 12 = \text{Right-hand side}$$

Example:

Solve the equation $3(x + 4) - 2(2x - 3) = -2(2x - 18)$.

Step 1: Remove the brackets, $\therefore 3x + 12 - 4x + 6 = -4x + 36$

Step 2: Collect the terms. $\therefore 3x - 4x + 4x = 36 - 12 - 6$

$$3x = 18$$
$$\therefore x = 6$$

Check: If $x = 6$

$$\begin{aligned}\text{Left-hand side} &= 3\,(6 + 4) - 2\,(12 - 3)\\ &= 30 - 18\\ &= 12\\ \text{Right-hand side} &= -\,2\,(12 - 18)\\ &= (-\,2) \times (-\,6)\\ &= 12\end{aligned}$$

EXERCISE 6G

Solve the equations and check your answers. Some of the answers are negative numbers.

1. $a + 5 = 0$

2. $2x + 12 = 4$

3. $5 = 7 + m$

4. $8 = 2 + 3f$

5. $11 = 6 + 2x$

6. $4 - 3x = 13$

7. $2 - 2x = 10$

8. $-\,3x - 5 = 10$

9. $x + 8 - 5x = 20$

10. $a - (3 - a) = 5$

11. $-\,(x + 2) = -\,4$

12. $2\,(4 + a) = 2$

13. $-\,3\,(2a - 5) = -\,9$

14. $3x = 56 - (x + 12)$

15. $4\,(t - 1) + 3\,(1 - 2t) = -\,(t + 4)$

16. $5\,(x + 1) + 2\,(x - 1) = -\,25$

17. $4\,(x - 2) - 3\,(3 - 2x) - 3 = 0$

18. $-\,2\,(2x - 9) - 5\,(7 + 3x) - 2 = 0$

19. I start with a number, add 15 and divide the result by 3. I find I am left with 2. What was the number?

20. When I add 7 to a number and multiply the result by 4, I get a total of 48. What is the number?

CHAPTER 7

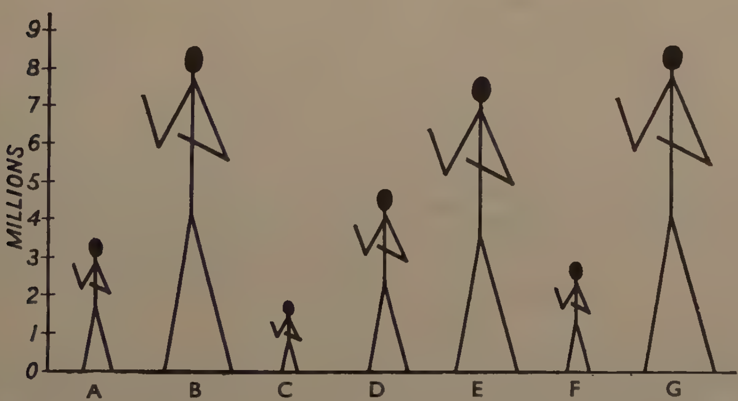
GRAPHS

GRAPHS ARE simply a means of showing numbers by pictures. It is much easier to take in the size of several pictures than to grasp the differences between a collection of numbers set out in a table. For example, here are the populations of several important cities.

City A	3 480 000	City E	7 800 000
City B	8 350 000	City F	2 850 000
City C	1 970 000	City G	8 530 000
City D	4 800 000		

Below are the same figures presented in picture form. A drawing of a man represents the inhabitants of each city, and the size of the drawing depends on the size of the population of that city.

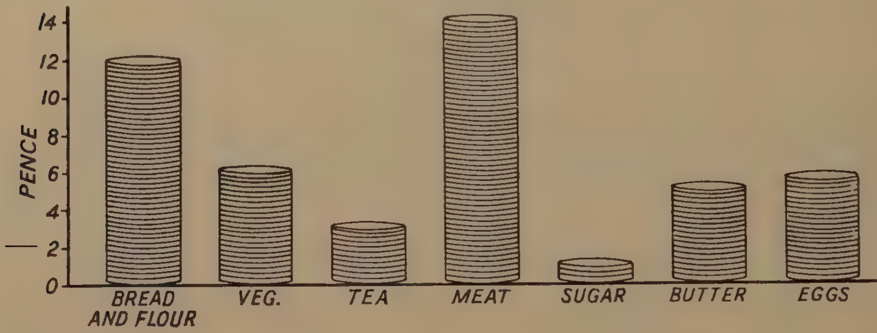
This graph shows at once the largest and the smallest populations among the cities we are considering (which are they?). But, of course, the pictures by themselves give no idea of the actual numbers: to do this we must produce a vertical *scale* on the left of the diagram, thus:



This makes the graph much more precise: we know that the populations are proportional to the size of each drawing, but what exactly do we mean by size? This graph shows that by size here we mean height, because that is the easiest quantity to measure.

EXERCISE 7A

The following graph shows the amount of money spent each day on various items of food by a single person.



1. On which item is most money spent?

2. How much is spent on tea?

3. How much is spent on sugar? How much on bread and flour?

4. Now draw your own graph to illustrate the following figures, which tell how much money was spent weekly on various items by an average British family in 1968.

Food	£6·59
Fuel	£1·55
Rent	£3·16
Clothing	£2·21
Transport	£3·27

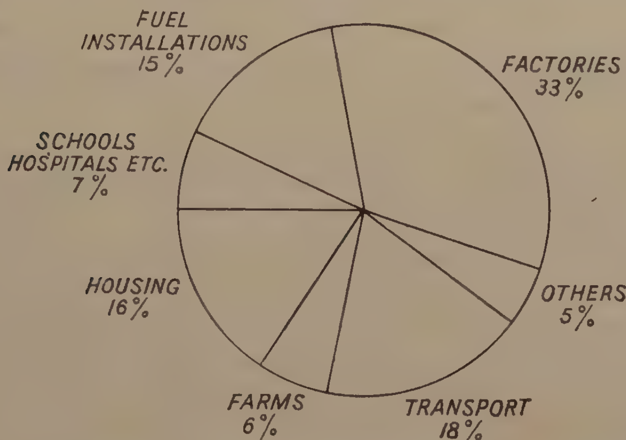
The Importance of Statistics

Collections of figures, like the populations of cities discussed above, are called STATISTICS. They are now used in all branches of daily life, but they originally arose from the wish of kings or governments to know what taxes they might raise from a particular area, or how many soldiers they might conscript to go to war. (Governments still use statistics to gain this knowledge.)

The Domesday Book compiled by the Normans was a collection of statistics, and it was a similar collection of tax statistics decreed by Caesar Augustus that brought Mary and Joseph to a crowded Bethlehem nearly two thousand years ago. Nowadays, of course, statistics are used by many people: by firms to find out if their products are satisfactory; by doctors to see if a new method of treating a disease is successful; by pools punters who study football results to help them to predict the results of future matches.

But statistics require expert handling, and can easily mislead. The makers of a new detergent, "Splurgo", can claim in their advertisements that it gives the whitest wash ever. This will persuade some people to buy it. They can then use statistics to show that more and more people are using Splurgo, thus "proving" that it gives the whitest wash ever.

There are other ways of presenting statistics. One such is called a pie-chart.



Here is a pie-chart to show how a certain country distributed its total investment in building among various items. Notice how the percentages are shown round the rim of the chart, as it is not possible to draw a scale as we did with our previous graphs. This sort of graph is useful to show how a given quantity (here, total investment) is divided up, like a pie at the dinner table, rather than to show actual amounts (the chart above does not tell you how much was actually spent on farms).

To construct such a chart in percentages, we first note that one per cent will be one-hundredth of the complete circle. Since there are 360 degrees altogether at the centre, one hundredth of this will be 3·6 degrees. Now the percentage invested in housing in the above chart is 16, so the angle of the wedge representing housing will be

$$16 \times 3{\cdot}6 \text{ degrees} = 57{\cdot}6 \text{ degrees}$$
$$\text{or } 58 \text{ degrees, roughly.}$$

(It is difficult, and needlessly fussy, to try to be more accurate with an ordinary protractor.)

In the same way we can work out the angles for the other wedges: this should be done *before* starting to draw the chart, so that we can check the accuracy of the arithmetic by adding up all the angles we have worked out. (What should be the total?)

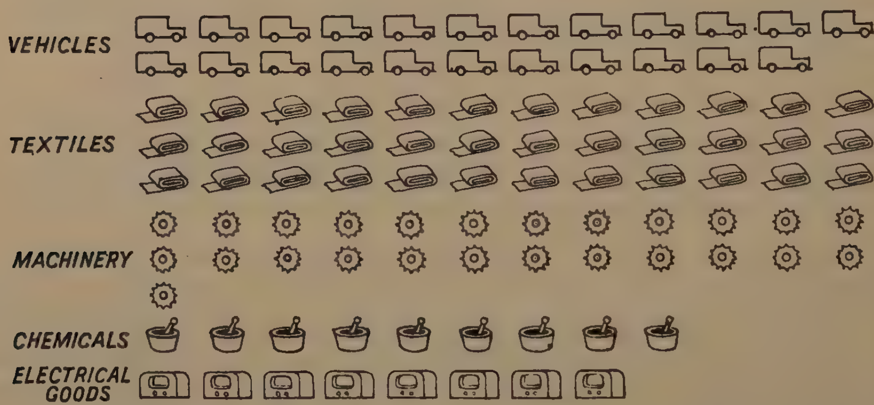
EXERCISE 7B

Make a pie-chart to illustrate the following figures, which show how the students at a certain university are distributed.

Science	40 per cent	History	8 per cent
Languages	14 per cent	Engineering	28 per cent
Others	10 per cent		

Another Kind of Chart

Yet another way of presenting figures is called the *ideograph*. Here is one to show how exports were distributed in a certain year.



The little symbols are called *isotypes*. Here each one represents one million pounds' worth of goods.

What value of vehicles was exported? Of textiles? Of electrical goods?

EXERCISE 7C

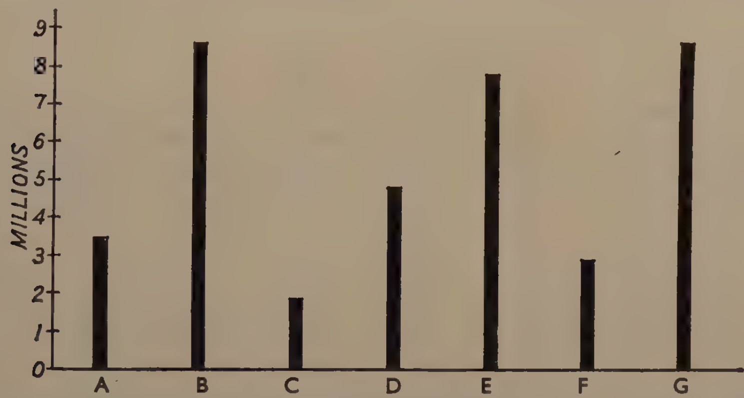
Draw an ideograph to illustrate a year's expenditure by one family from the following figures. Choose your own isotypes to represent the various items.

Food	£350
Clothes	£150
Rent and rates	£150
Repairs	£50
Entertainment and holidays	£100
Travelling	£100
Other items	£200

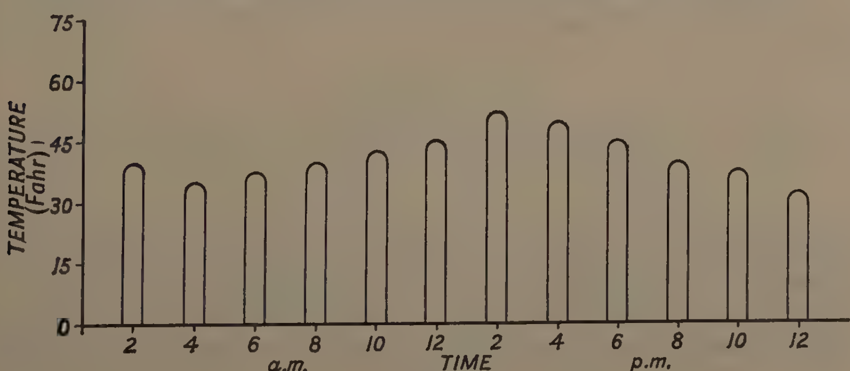
Let each of your isotypes represent £50 spent on that particular item.

Column Graphs

Let us now return to the picture graph (page 211). The little pictures of men remind us that the graph is about the numbers of people in various cities: but the title of the graph tells us that anyway. The important *numerical* information is given by the height of each man, which tells us, via the vertical scale, the number of inhabitants of each city. This could be equally well achieved by scrapping the pictures of men and simply drawing vertical columns, thus:



Such a graph is called a column graph, and gives just as much information as a picture graph, if less decoratively. The information about the population of each city is given by the *height* of the column; the width and the area are not important, but it would be misleading to have the columns of different widths, so we always try in a column graph to keep the width of each column the same. A thin pencil line is enough, provided we can see it. Here is another column graph, showing the temperature measured at different times during one day in April. (You might take each column as a picture of the column of mercury in the thermometer from which the readings were taken.)



This graph differs in one important respect from the others. It shows the *same* quantity (temperature) measured at different times.

What was the temperature at 4 a.m.? At 10 a.m.? At 6 p.m.?

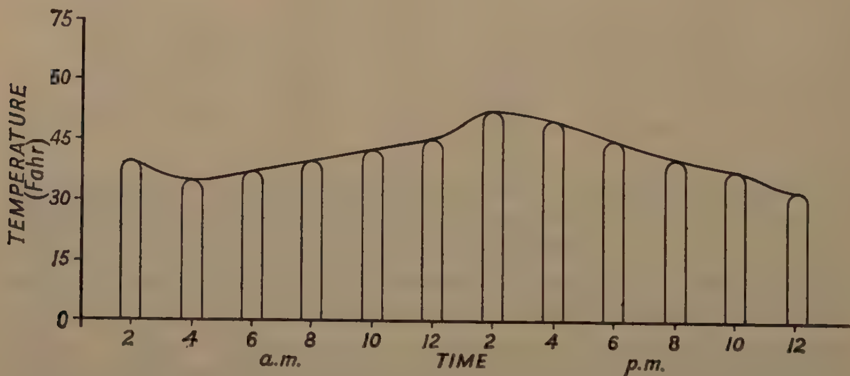
Although the measurements are taken every two hours, we know that the temperature is continually changing, so that we can guess the temperature at 9 a.m. by imagining the probable height of the column between 8 a.m. and 10 a.m. if one were drawn. What is your guess?

What was the highest temperature of the day?

At what time did it occur?

Can you be sure of your answers?

The process of guessing the temperature at 9 a.m. by comparison with the heights of the columns at 8 a.m. and 10 a.m. is called *interpolation*. It is much easier to interpolate if we join up the tops of the columns with a single curved line, thus:



Of course, we cannot guarantee that the curve accurately represents the temperatures between the measured times, but on the whole it will probably give us a good estimate. What was the probable temperature at 1 p.m.? At 5.30 p.m.?

EXERCISE 7D

1. Draw a column graph to illustrate a boy's marks in a series of practice tests before taking an examination. The marks were as follows:

35 42 34 54 62 21 56 68

During one test he wasn't feeling well. Which one do you think it was? On the whole, did the practice tests make any difference?

2. In an experiment, a number of different weights were hung on the end of a spring, and on each occasion the length of the spring was measured.

Here are the results:

Weight in kg	2	3	4	5	6	7	8
Length in cm	15	16·5	18·1	19·4	20	22·5	23·9

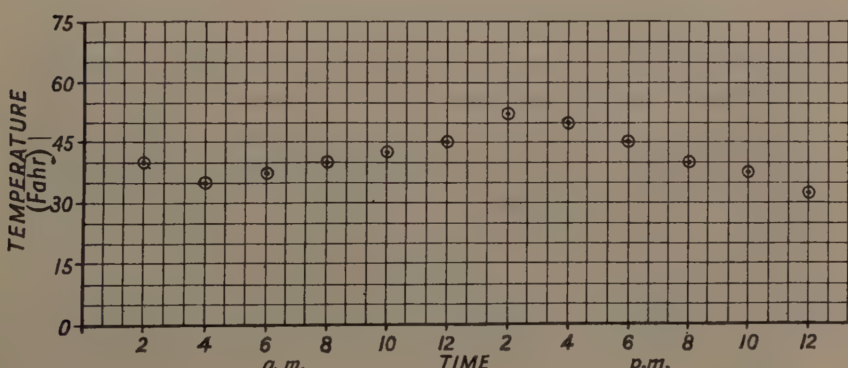
Draw a column graph to illustrate the results.

A mistake was made in one of the measurements. Which one do you think it was? Leaving out the wrong measurement, join up the tops of the columns with a line, and use this line to estimate: (a) How long the spring would be with a weight of 4·5 kg hanging on it; (b) What the wrong measurement should have been; (c) What weight do you think would make the spring exactly 17 cm long?

Graphs and their Axes

For most graphical work it is very useful to use squared paper, as this saves a great deal of measuring with a ruler. Such paper (often called graph paper) is divided into squares of dimensions 1 in or 1 cm, which are themselves sub-divided into smaller squares of dimensions $\frac{1}{10}$ in or 1 mm.

To use such paper for the temperature graph on page 215 we first draw two heavy lines along which we will mark the scales. These are called the *axes*. The vertical one is called the *temperature axis* and the horizontal one is called the *time axis*. After putting in the scales it is no longer necessary to draw each column: we simply mark in the top of each column with a small dot or cross. (If a dot is used it is usual to put a small circle round it, otherwise it will be difficult to see.)



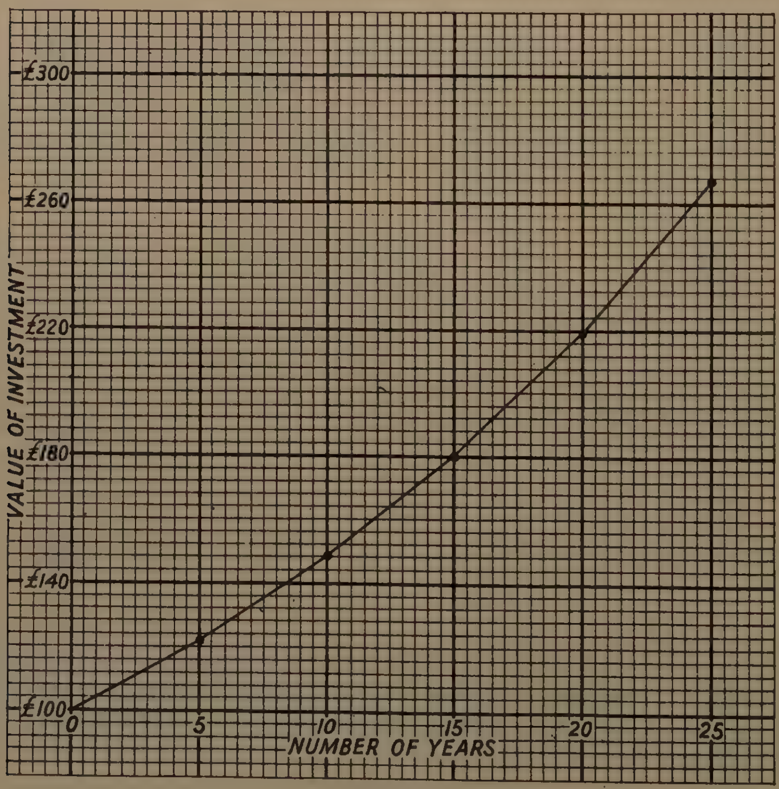
Notice how the lines of the squared paper help in the problem of interpolation: it is now much easier to guess the probable temperature at various intermediate times. What is your guess for the temperature at 11 a.m.? At 12.30 p.m.?

Here is a table of figures showing how much money you will have if you invest £100 at 4 per cent compound interest and leave it for the stated times.

Number of years	0	5	10	15	20	25
Amount in £	100	122	148	180	219	267

Now the graph we draw will depend on the size of the piece of graph paper we have. Let us suppose that our paper is 6 in square. We can draw our horizontal time axis along the lowest heavy line, and our vertical money axis along the heavy line nearest the left-hand side of the paper. If we choose 1 in to represent five years, the time scale will fit nicely, as our largest figure, 25 years, will lie 5 in along the axis. Now the *lowest* figure on the money axis is 100, so it will save space if we start measuring our money scale, not from zero, but from £100.

If we now choose 1 in to represent £40 on this axis, the 5 in we have left on our paper will carry us from £100 to £300, which conveniently covers the range of numbers that we want. We can now complete the graph, which will look like this:



Misleading Scales

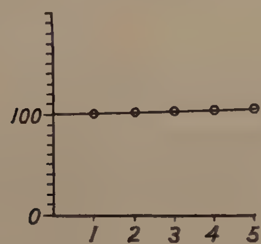
The business of choosing scales when you are drawing a graph is very important, in order to use your paper to the best advantage. It is equally important to examine the scales carefully when you are studying a graph. Some advertisers, to boost their products, pretend to show "scientific" graphs which have no scale on at all, and even when a scale is shown, it is easy to arrange it so as to mislead those who know very little about graphs.

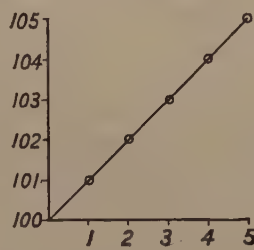
Example:

During the five years that a certain political party was in power, the "cost-of-living" index went up as follows:

Year	1	2	3	4	5
Cost-of-living Index	101	102	103	104	105

During the next General Election, the Government and Opposition each produced a graph of these figures. The Government's graph is on the left; the Opposition's, using the same information, is on the right:





You can clearly see how the Government concealed the rise in the cost-of-living index, by measuring from zero and choosing a small scale; and how the Opposition emphasized the rise by measuring from 100 and choosing a large scale. The same set of figures can be made to look quite different in this way. The scales are very important. When next you see a graph in a television or newspaper advertisement, examine it carefully and see what it really means—or whether it means anything at all.

EXERCISE 7E

1. A car was tested at various speeds to discover within what distance it could be brought to a standstill. The results are given in the following table:

Speed in mile/h	10	20	30	40	50	60
Distance in yd	5	14	27	44	65	90

Draw a graph to illustrate these figures. Remember to choose your scales carefully before you start. Estimate the distance required to stop the car from a speed of 35 mile/h. From about what speed would it take 50 yd to stop?

2. A boy sets out on a bicycle at 9 a.m. and rides at a steady speed of 9 mile/h until he reaches a friend's house $22\frac{1}{2}$ miles away. How long does the journey take him? Make out a table to show how far he has travelled after various times at intervals of half-hour. Here is the table started for you:

Time	9.00	9.30	10.00	10.30	11.00	11.30
Distance travelled in miles	0	4·5	9			

Plot the points on a graph and join them up with a ruler (this will be possible because the points should lie on a straight line). Use your graph to estimate how far he has gone: (a) at 10.15; (b) at 11.10.

(You can check your answers by arithmetic if you like.) At what time will he be 10 miles from home?

This sort of graph, illustrating a journey, is called a *travel graph*.

3. A gun fires a shell with a muzzle velocity of 800 feet per second. The height of the shell after various times is given by the following table:

Time in seconds	0	5	10	15	20	25
Height in feet	0	1600	2400	2400	1600	0

Draw the graph of these figures and use it to answer the following questions:

(a) What is the height of the shell after 8 seconds?

(b) What is the greatest height of the shell given in the table?
Was this the greatest height the shell reached?
What do you think was the greatest height the shell reached?

(c) At what times was the shell at a height of 2000 ft? Why are there two answers?

Another Kind of Graph

In a certain examination there were 81 candidates, and a complete list of the results as percentages is given in the following table:

88	72	70	63	56	54	51	45	30
80	72	69	62	56	54	50	44	26
78	72	68	62	56	53	49	41	25
78	71	67	61	55	53	49	41	24
78	71	65	61	55	53	49	41	24
78	71	64	61	55	52	48	41	20
77	71	64	60	55	52	47	37	17
77	70	63	60	54	52	46	34	15
76	70	63	58	54	51	46	34	7

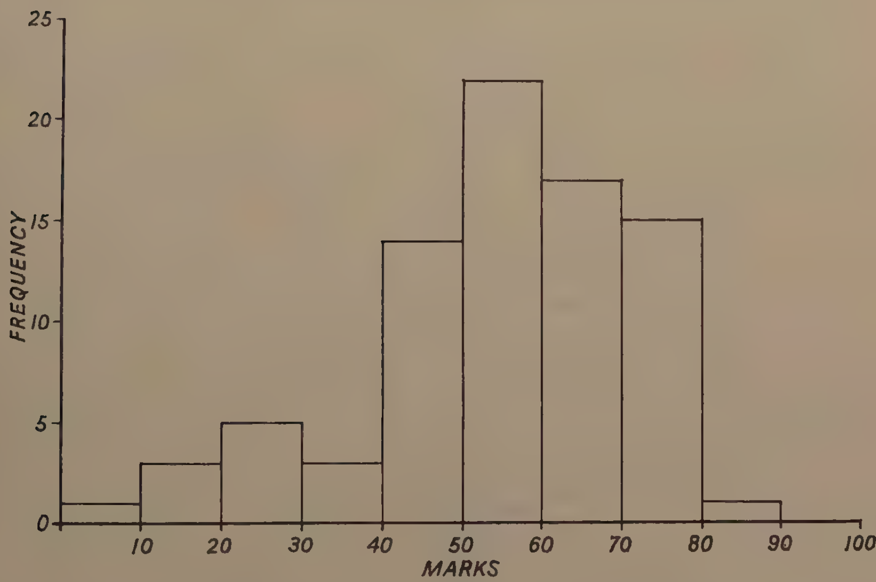
Laid out in a table like this, the above is little more than a rather overwhelming collection of figures, so let us now analyse them a little more closely. To do this, we count how many of the figures lie within a certain range for instance, between 41 and 50 *inclusive* there are 14 figures, as you

can check by counting them in the table above. In the same way we count how many figures lie within other ranges, and lay out our results in a fresh table like this:

0-10	1	51-60	22
11-20	3	61-70	17
21-30	5	71-80	15
31-40	3	81-90	1
41-50	14		

We now have a table showing us how many marks lie within each range. You can see that a mark between 51 and 60 occurs 22 times, which is more often than in any of the other ranges. The number 22 is called the *frequency* with which marks 51-60 occur. In the same way, the frequency of the marks 11-20 is 3. What is the frequency of the marks 71-80?

We can now illustrate these frequencies graphically, as follows:



Notice how each frequency is represented, not by a single point, but by a rectangle, whose width is the so-called class-range (e.g. 41-50) and whose height is the frequency of that class range. This graph is called a *histogram.* It shows how a set of figures is distributed. In this histogram we can see that most of the marks are packed up in the top half, roughly, between 41 and 80. It was quite a well-set examination: not too easy (or else nearly everybody would have got 80 or 90, and it would hardly have been worth doing) but easy enough for most people to get quite respectable marks.

EXERCISE 7F

1. One year in a certain school 104 candidates sat for the O-Level of the G.C.E. Four candidates failed to pass in any subject at all; three passed in one subject only. This and similar information is given in the following table:

Number of subjects passed	0	1	2	3	4	5	6	7	8
Number of candidates	4	3	5	7	16	44	15	5	3

Draw a histogram to illustrate these results.

2. Collect the heights of as many as possible of your friends and relatives and make out a table of frequencies with a class-interval of 2 in, i.e. How many there are between 4 ft and 4 ft 2 in; how many between 4 ft 2 in and 4 ft 4 in, etc. Then draw the histogram of your figures. (If you cannot decide whether someone is under 4 ft 4 in or over it, give him the benefit of the doubt and count him as over 4 ft 4 in. It is not important which class you decide to place him in, so long as you treat all doubtful cases in the same way.)

CHAPTER 8

FRACTIONS AND COMPOUND BRACKETS

IN CHAPTER 2 we learnt that the contents of a bracket are to be treated as a single number. A number which has two parts is called a BINOMIAL. Thus $(a + 3)$, $(x^2 + 5)$, $(3x + 2y)$ are all binomials. You have also had some practice in removing and inserting brackets. If a binomial, say $(x + y)$, is multiplied by n, we can say

$$n(x + y) = nx + ny$$

Similarly, $n(x - y) = nx - ny$

By using brackets we can combine different expressions and make calculation much easier. A very simple example would be:

$$8^2 + 8 \times 4$$

This is equivalent to $(8 \times 8) + (8 \times 4)$ and is of the type $nx + ny$, above. We could therefore write:

$$8(8 + 4) = 96$$

In the same way,

$$\begin{aligned} 18 \times 11 - 18 &= 18 \times 11 - 18 \times 1 \\ &= 18(11 - 1) \\ &= 180 \\ 70 \times 69 &= 70(70 - 1) \\ &= 4900 - 70 \\ &= 4830 \\ \text{and } 26 \times 1\tfrac{3}{4} &= 26(2 - \tfrac{1}{4}) \\ &= 52 - 6\tfrac{1}{2} \\ &= 45\tfrac{1}{2} \end{aligned}$$

EXERCISE 8A

Use the two rules above, $n(x + y) = nx + ny$
and $n(x - y) = nx - ny$, to evaluate the following:

1. $5 \times 4 + 5 \times 6$

2. $11 \times 6 + 11^2$

3. $5 \times 2{\cdot}3 + 5 \times 3{\cdot}7$

4. $9 \times 8 - 9 \times 2$

5. $6^2 - 6 \times 2$

6. $16 \times 1{\cdot}6 - 16 \times 1{\cdot}1$

7. $7 \times 4{\cdot}3 + 7 \times 3{\cdot}7$

8. $17 \times 9 + 17$

9. $23 \times 12 - 23 \times 2$

10. $4 \times 2{\cdot}7 + 4 \times 3{\cdot}8$

11. $a \times 14 + a \times 3$

12. $x \times 3m + x \times 2m$

13. $2a \times 3y + 2a \times 2b$

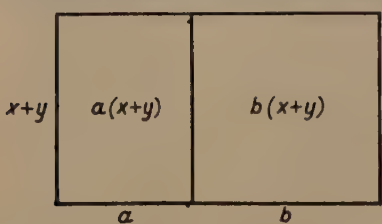
14. $5x \times 2y + 5x \times 3z$

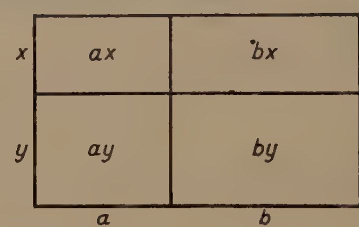
Identities and Expansion

$n(x + y) = nx + ny$ is an example of an IDENTITY. Each side of the equation is an expanded or simplified form of the other.

$$\frac{a}{3} + \frac{a}{4} = \frac{7a}{12}$$

is another identity. Some of the most important identities are formed by multiplying together two binomials. Consider these rectangles:





They are both diagrams of the same rectangle $(a + b)$ in long and $(x + y)$ in wide. So the area must be $(a + b)(x + y)$ in². The first rectangle is divided into two parts, and the total area is $a(x + y) + b(x + y)$ in².

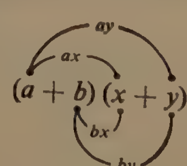
The second rectangle is divided into four parts and the total area is

$$ax + ay + bx + by \text{ in}^2$$

Evidently $(a + b)(x + y) = a(x + y) + b(x + y)$

$$= ax + ay + bx + by$$

This shows us the rule for multiplying two binomials. Each term in the second bracket must be multiplied by each in the first. Diagrammatically, the process would look like this:



$ax + ay + bx + by$ is an EXPANSION of $(a + b)(x + y)$.

Example: Expand $(2p + q)(a + 4b)$.

$$2p \times a = 2ap,$$
$$2p \times 4b = 8bp,$$
$$q \times a = aq,$$
$$q \times 4b = 4bq,$$
$$\therefore (2p + q)(a + 4b) = 2ap + 8bp + aq + 4bq$$

(No further simplification is possible, as the terms are unlike.)

EXERCISE 8B

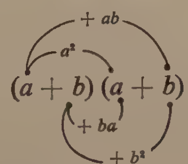
Expand the following:

1. $(a + 1)(b + 2)$
2. $(x + 3)(y + 4)$
3. $(5 + n)(3 + m)$
4. $(12 + c)(8 + d)$
5. $(f + 3)(g - 2)$
6. $(3 + f)(2 - g)$
7. $(x - 7)(y - 2)$
8. $(4x + 3)(5x + 13)$
9. $(x + 5)(x + 8)$
10. $(a + 2\frac{1}{2})(a + 4)$
11. $(a - 7)(a + 7)$
12. $(5 + x)(5 - x)$
13. $(x + 4)(x + 4)$
14. $(a - 3)(a - 3)$
15. $(4p - 2)(4p - 2)$
16. $(3x + 7)(3x + 7)$
17. $(7x + y)(2x + y)$
18. $(x + y)(x + y)$
19. $(x - y)(x - y)$
20. $(x - y)(x + y)$

Other Forms of Identity

Example: Work out $(a + b)^2$.

This means $(a + b) \times (a + b)$, and is *not* the same as $a^2 + b^2$.



$$= a^2 + ab + ab + b^2$$
$$= a^2 + 2ab + b^2$$

Similarly: $(a - b)^2 = a^2 - 2ab + b^2$ and $(a - b)(a + b) = a^2 - b^2$

These are three important examples of identities and should be learnt. If you had to calculate the area of a path 2 yd wide around a square lawn, side 32 yd, could you do it by using the last of these identities?

Square Roots

In the chapter on directed numbers, we learnt that x^2 can mean

$$(+x) \times (+x) \text{ or } (-x) \times (-x)$$

So the square root of x^2 can be either $+x$ or $-x$.

Both of these can be written together, $\pm x$ ("plus or minus x").

$$\text{So if } x^2 = 16$$
$$x = \pm 4.$$

But the sign $\sqrt{}$ is used only for the positive square root:

$$\text{and}\quad \sqrt{x^2} = +x$$
$$\sqrt{16} = +4.$$

Can there be a square root for a negative number?

Now if we have an equation with a binomial squared, say, $(x + 3)^2 = 36$, what is the value of x?

If the square is equal to 36, the square root $(x + 3)$, must be ± 6.

$$\begin{aligned} \text{In other words } (x + 3) &= +6 \\ \text{or } (x + 3) &= -6 \\ \text{If } x + 3 &= +6 \\ \text{then } x &= 3 \\ \text{if } x + 3 &= -6 \\ \text{then } x &= -9 \end{aligned}$$

In other words there may be two possible values for x in an equation of this type.

EXERCISE 8C

Find the values of x in the following examples:

1. $(x + 2)^2 = 9$
2. $(x - 4)^2 = 25$
3. $(x + 7)^2 = 36$
4. $(x - 1)^2 = 4$
5. $(x - 1)^2 = 81$
6. $(3x + 2)^2 = 16$
7. $\sqrt{(x + 3)} = 9$ [If $\sqrt{(x + 3)} = 9$, what does $(x + 3)$ equal?]
8. $\sqrt{(2x - 2)} = 4$
9. $5 = \sqrt{(3x + 4)}$
10. $(3x + 11)^2 = 64$

Binomials in Fractions

You already know how to simplify fractions: $\frac{x}{5} + \frac{x}{3} = \frac{3x + 5x}{15}$

$$= \frac{8x}{15}$$

It is important to remember what happens when you do this. We look for the L.C.M. of the denominators, in this case 15. The denominator of $\frac{x}{5}$ is multiplied by 3, to increase it to the L.C.M. and the numerator has also to be multiplied by 3. In the same way $\frac{x}{3} = \frac{5x}{15}$. When the terms have a common denominator they can be added. Exactly the same method is used when the numerator is a binomial. Consider this example:

$$\frac{x+3}{4} + \frac{x+2}{5}$$

The line between numerator and denominator acts as a bracket, and $\frac{(x + 3)}{4}$ is equal to $\frac{x + 3}{4}$.

The L.C.M. is 20 and the denominator of $\frac{x+3}{4}$ must be multiplied by 5.

$\therefore$ the numerator must also be multiplied by 5.

$$\therefore \frac{x+3}{4} = \frac{5(x+3)}{20}$$

$$\text{and } \frac{x+2}{5} = \frac{4(x+2)}{20}$$

$$\text{so } \frac{x+3}{4} + \frac{x+2}{5} = \frac{5(x+3)}{20} + \frac{4(x+2)}{20}$$

$$= \frac{5x + 15 + 4x + 8}{20}, \text{ removing the brackets,}$$

$$= \frac{9x + 23}{20}$$

Example: Simplify: $\dfrac{a+3b}{2b} - \dfrac{a-4b}{5a}$

The L.C.M. of $2b$ and $5a$ is $10ab$.

$\therefore$ the first denominator, $2b$, must be multiplied by $5a$, giving

$$\frac{a+3b}{2b} = \frac{5a(a+3b)}{10ab}$$

The second denominator must be multiplied by $2b$, giving

$$\frac{a-4b}{5a} = \frac{2b(a-4b)}{10ab}$$

$$\therefore \frac{a+3b}{2b} - \frac{a-4b}{5a} = \frac{5a(a+3b) - 2b(a-4b)}{10ab}$$

which, clearing the brackets, gives $\dfrac{5a^2 + 15ab - 2ab + 8b^2}{10ab}$ (a minus sign outside the bracket changes the signs inside)

$$= \frac{5a^2 + 13ab + 8b^2}{10ab}$$

Check: Take any simple value for a and b; let $a = 2$, $b = 3$.

Then: $$\frac{a+3b}{2b} - \frac{a-4b}{5a} = \frac{2+9}{6} - \frac{2-12}{10}$$

$$= \frac{11}{6} - \frac{-10}{10}$$

$$= \frac{110+60}{60} = \frac{170}{60} = 2\tfrac{5}{6}$$

$$\frac{5a^2 + 13ab + 8b^2}{10ab} = \frac{20 + 78 + 72}{60}$$

$$= \frac{170}{60} = 2\tfrac{5}{6}$$

Such a check by substitution does not prove the answer correct, except for those particular values chosen, but does point to the probability that you have worked out the answer correctly.

EXERCISE 8D

Simplify the following. Notice that in No. 9, for example,

$$\tfrac{1}{2}(4x - 3) = \frac{4x - 3}{2}.$$

1. $\dfrac{a+1}{2} + \dfrac{a+2}{3}$
2. $\dfrac{b+5}{5} + \dfrac{b+7}{3}$
3. $\dfrac{x+3}{7} + \dfrac{x-2}{6}$
4. $\dfrac{2x+7}{2} - \dfrac{3x+4}{5}$
5. $\dfrac{3a+4}{4} - \dfrac{2a-1}{6}$
6. $\dfrac{a-7}{9} - \dfrac{5a-6}{8}$
7. $\dfrac{2n-8}{4} + \dfrac{3n-1}{8}$
8. $\dfrac{3m+13}{12} - \dfrac{7m-4}{6}$
9. $\tfrac{1}{2}(x+7) + \tfrac{1}{4}(x+2)$
10. $\tfrac{1}{5}(4x+3) + \tfrac{1}{3}(2x-8)$
11. $\tfrac{2}{3}(x+6) - \tfrac{4}{5}(x-1)$
12. $\tfrac{6}{5}(2b-7) - \tfrac{2}{3}(5b-2)$
13. $\dfrac{a+2b}{5} - \dfrac{3a-b}{7}$
14. $\dfrac{4e-5f}{9} + \dfrac{3e+8f}{5}$
15. $\dfrac{a+b}{c} + \dfrac{a-b}{3c}$
16. $\dfrac{2x-y}{z} - \dfrac{3x+5y}{3z}$
17. $\dfrac{4a-b}{c} + 1$
18. $\dfrac{x-y}{5n} - \dfrac{x+y}{7n}$
19. $\dfrac{3c-d}{2c} + \dfrac{c+4d}{5d}$
20. $\dfrac{2x-7y}{xy} + \dfrac{3x+y}{2y}$
21. $\dfrac{8g+7h}{n} - \dfrac{2g-2h}{gh}$
22. $\dfrac{a^2+7}{b} + \dfrac{2a^2+10b}{b^2}$

Solution of Equations

There are many equations which involve fractions, both simple and with binomial numerators. Almost the first written mathematical problem, on an Egyptian papyrus, records an equation which, using our symbols, would look like this:

$$n + \frac{n}{7} = 19$$

Example:

Here is an equation with simple fractions:

$$\frac{a}{2} - \frac{2a}{5} = \frac{1}{2}$$

The first step is to change each fraction so that all have the same denominator. What is the lowest number we can use? It must be the L.C.M. of the denominators, 10.

The first term $\frac{a}{2}$ becomes $\frac{5a}{10}$

The second $\frac{2a}{5}$ becomes $\frac{4a}{10}$

and $\frac{1}{2}$ becomes $\frac{5}{10}$

We can now write $\frac{5a}{10} - \frac{4a}{10} = \frac{5}{10}$

If we multiply each term of both sides by the denominator, we can clear the fractions by cancelling:

$$\cancel{10} \times \frac{5a}{\cancel{10}} - \cancel{10} \times \frac{4a}{\cancel{10}} = \cancel{10} \times \frac{5}{\cancel{10}}$$

$$\text{leaving } 5a - 4a = 5$$

$$\therefore \quad a = 5$$

Check: L.H.S. $\frac{5}{2} - \frac{10}{5} = \frac{1}{2}$ = R.H.S.

In effect this amounts to multiplying each term by the L.C.M., and we could write $\frac{10a}{2} - \frac{20a}{5} = \frac{10}{2}$, which, when cancelled, leaves us with $5a - 4a = 5$.

This process can be much shortened.

Example: $\frac{3a}{8} - \frac{1}{2} = \frac{a}{6}$

The mental process is as follows:

1. The L.C.M. is 24
2. $24 \div 8 = 3$,
3. $3 \times 3a = \ldots 9a$,
4. $24 \div 2 = 12$,
5. $12 \times 1 = \ldots 12$,
6. $24 \div 6 = 4$,
7. $4 \times a = \ldots 4a$.

The equation can now be written: $9a - 12 = 4a$

$$\therefore 9a - 4a = 12$$

$$\therefore 5a = 12$$

$$\therefore a = \frac{12}{5}$$

$$= 2\tfrac{2}{5} \text{ or } 2{\cdot}4$$

Check substituting 2·4 for a,

$$\text{L.H.S. } \frac{7{\cdot}2}{8} - \frac{1}{2}$$

$$= \frac{72}{80} - \frac{40}{80}$$

$$= \frac{32}{80}$$

$$= \frac{4}{10} \text{ or } {\cdot}4$$

$$\text{R.H.S. } \frac{2{\cdot}4}{6} = \frac{24}{60}$$

$$= \frac{4}{10} \text{ or } {\cdot}4$$

$$\therefore a = 2{\cdot}4 \text{ is a solution of the equation.}$$

Fractions having unknown denominators are treated in the same way:

Example: $\frac{1}{a} - \frac{1}{2a} = \frac{1}{5}$

L.C.M. of a, $2a$, and 5 is $10a$

$$10a \div a = 10$$
$$10 \times 1 = \ldots 10$$
$$10a \div 2a = 5$$
$$5 \times 1 = \ldots 5$$
$$10a \div 5 = 2a$$
$$2a \times 1 = \ldots 2a$$

The equation can now be rewritten:

$$10 - 5 = 2a$$
$$\therefore 5 = 2a$$
$$\therefore a = 2\tfrac{1}{2}$$

Check: L.H.S. $\frac{1}{2\frac{1}{2}} - \frac{1}{5} = \frac{2}{5} - \frac{1}{5}$

$$= \frac{1}{5} = \text{R.H.S.}$$

Do exactly the same thing for equations with binomial numerators:

Example: $\frac{x+2}{10} + \frac{x+3}{5} = \frac{7}{8}$

The L.C.M. of 10, 5 and 8 is 40. We want all the terms to have the same denominator.

$$\therefore \frac{x+2}{10} = \frac{4\,(x+2)}{40}, \quad \frac{x+3}{5} = \frac{8\,(x+3)}{40}, \quad \text{and } \frac{7}{8} = \frac{5 \times 7}{40}$$

Then $\frac{4\,(x+2)}{40} + \frac{8\,(x+3)}{40} = \frac{5 \times 7}{40}$

Multiplying throughout by 40,

$$4\,(x+2) + 8\,(x+3) = 35$$
$$\therefore 4x + 8 + 8x + 24 = 35$$
$$\therefore 4x + 8x = 35 - 8 - 24$$
$$\therefore 12x = 3$$
$$\therefore x = \frac{3}{12}$$
$$= \frac{1}{4}$$

Check: If $x = \frac{1}{4}$

$$\text{L.H.S.} = \frac{\frac{1}{4}+2}{10} + \frac{\frac{1}{4}+3}{5}$$
$$= \frac{2\frac{1}{4}}{10} + \frac{3\frac{1}{4}}{5}$$
$$= \frac{9}{40} + \frac{13}{20}$$
$$= \frac{9+26}{40}$$
$$= \frac{35}{40} = \frac{7}{8} = \text{R.H.S.}$$

EXERCISE 8E

Solve the equations and check your solutions:

1. $\frac{1}{3a} + \frac{1}{2a} = \frac{1}{6}$
2. $\frac{1}{n} + \frac{1}{2n} = \frac{3}{8}$
3. $\frac{1}{2b} - \frac{1}{3b} = \frac{1}{4b} - 1$
4. $\frac{3a}{5} + \frac{2a}{8} = \frac{1}{2}$
5. $\frac{c}{4} - \frac{2c}{12} = \frac{2}{24}$
6. $\frac{1}{3} + \frac{5}{n} = \frac{7}{n}$
7. $2n + \frac{7n}{11} = \frac{5}{11}$ (Hint: $2n = \frac{2n}{1}$)
8. $\frac{a}{2} = \frac{4a}{3} - 1$
9. $\frac{x}{3} - \frac{x}{9} = 4$
10. $\frac{4c}{3} - 2 = \frac{c}{4}$
11. $\frac{1}{4} - n = 3n + \frac{5}{4}$
12. $2\frac{2}{3}x - 2\frac{1}{4}x = 1$ (Hint: $2\frac{2}{3}x = \frac{8x}{3}$)
13. $\frac{a}{0{\cdot}7} = 5$
14. I am x years old and my son is one-third of my age. Our combined ages total 60. How old am I?
15. I spend $\frac{1}{4}$ of my savings on a model yacht, and $\frac{1}{12}$ on a cricket bat. When I count what is left I find I have £8. How much was there at first?
16. $1 + \frac{5a}{2} = a + 3$
17. $\frac{n}{3} \div 1\frac{2}{3} = \frac{2}{5}$
18. $\frac{3n+2}{2} + \frac{4n-9}{3} = 2$
19. $\frac{y+2}{5} - \frac{y+2}{7} = 3y - 2$
20. $\frac{5-2y}{3} + \frac{3+3y}{4} = \frac{7y-2}{2}$
21. $x - 1 = \frac{2x+3}{5} - \frac{3x-8}{7}$
22. $\frac{x+1}{2} + \frac{x-1}{3} = \frac{7}{3}$
23. $\frac{2a-3}{5} - \frac{3a-7}{8} = \frac{4a+6}{2}$
24. $\frac{1}{4}(2n-1) + \frac{1}{6}(3n+1) = n + 1$ $\left[\text{Hint: } \frac{1}{4}(2n-1) = \frac{2n-1}{4}\right]$

25. $\dfrac{2x-3}{6} - \dfrac{2x-2}{3} + 11x = 0$ 26. $\dfrac{n}{-} + \dfrac{n}{7} = 19$

Compound Brackets

Consider this binomial, $d + 3h$. Let us suppose we want to multiply it by 32. This is fairly straightforward. We have then $32\,(d + 3h)$. But suppose now we want the square root of all of it. It won't do to write $\sqrt{}32\,(d + 3h)$. This means, find the square root of 32, and then multiply the result by $(d + 3h)$, whereas we want the square root of the *product* of 32 and $(d + 3h)$. We could write $\sqrt{}(32\,(d + 3h))$. But to avoid mistakes in a complicated expression like this, we usually use different kinds of brackets. As well as ( ), we use { }, [ ], or the vinculum, $\overline{\quad}$. So this expression could be written more clearly as $\sqrt{}\{32\,(d + 3h)\}$,

or $\sqrt{}[32\,(d + 3h)]$,

or $\sqrt{}(32 \times \overline{d + 3h})$.

This in fact is the formula for the speed of a wave, in feet per second, where $d =$ the depth of the water in feet, and h the height of the wave in feet. What is the speed of a wave 9 ft high moving across water 5 ft deep?

If it is necessary to simplify such an expression, begin by removing the *inner* brackets first. Thus:

$4x - \{3x - 4\,(2 - x)\} = 4x - \{3x - 8 + 4x\}$, removing the inner bracket.

$= 4x - 3x + 8 - 4x$, removing the outer bracket, and remembering that a minus sign outside the bracket changes all the signs within when the bracket is removed.

$= -\,3x + 8$

Example:

Subtract $2 + 3\,(n - 1)$ from $5n - 8\,(3n + 4)$.

Using a system of brackets we could rewrite this in the following way:

$$\{5n - 8\,(3n + 4)\} - \{2 + 3\,(n - 1)\}$$

which, removing the inner brackets $= \{5n - 24n - 32\} - \{2 + 3n - 3\}$

$= \{-\,19n - 32\} - \{3n - 1\}$

$= -\,19n - 32 - 3n + 1$

$= -\,21n - 31$

Example:

Simplify $\frac{1}{2}\,\{2\,(x - y) - 5\,(x + 2y)\}$

$= \frac{1}{2}\,\{2x - 2y - 5x - 10y\}$

Multiplying each term by $\frac{1}{2}$ $= x - y - \dfrac{5x}{2} - 5y$

Collecting the terms, $= -\,\dfrac{3x}{2} - 6y$

Example:

Solve the equation $2a - (3a + 4) + [(2a - 1) - (3 - 6a)] = 13$

Removing the brackets first, $2a - (3a + 4) + [2a - 1 - 3 + 6a] = 13$

Two steps are necessary. $\therefore 2a - 3a - 4 + 2a - 1 - 3 + 6a = 13$

Collect the terms,
$$\therefore 7a - 8 = 13$$
$$\therefore 7a = 21$$
$$\therefore a = 3$$

EXERCISE 8F

Nos. 1-14 simplify the expressions:

1. $3m - \{4 - (3m + 5)\}$

2. $6y + 3\{2(3y + 5)\}$

3. $7\{a + 2(3 + 2a)\}$

4. $2[(a + b)(a - b)]$

5. $6 + 3y[3(2 + y)]$

6. $b + 5\{b(b + 3)\}$

7. $\{(c - 4) - (-2c - 6)\} - 8$

8. $2e + [(e + 2f) - (e - f)]$

9. $n\{n(n + 4) - (n - 3)\}$

10. $3\{x + y(x + y)\} - 2\{x(x + y)\}$

11. $\frac{1}{2}\{3(x + 2y) + 4(3x + 5y)\}$

12. $-[5a - b + c]$

13. $1 + \{x^3 - (-x)^3\}$

14. $2(a + b) - \{2(a + b) - 2(a - b)\}$

In Nos. 15-24 find the value of the expressions if $a = 3$, $b = 4$, $c = \frac{1}{2}$, and $d = 6$.

15. $a\{(b + a) - (2c + d)\}$

16. $5c - 1 + c[7 + \sqrt{(a + d)}]$

17. $3\left[\frac{1}{7} - \frac{1}{a(d - a)}\right]$

18. $\frac{a}{2}\left\{3b + (d + 5) + 2c(a - b)\right\}$

19. $(ab)^2 + [4c - b(3 + d)]$

20. $\frac{b}{3}\left\{5 + \sqrt{(7 + 3a)}\right\}$

21. $\dfrac{(-a) \times (-b)}{(-c) \times (-d)}$

22. $6c\{d + (b + 4c)^2\}$

23. $\{5(1 - a) - 5a(1 - a)\} + 3$

24. $-5 + \sqrt{\{a^2 - (a + d)\}}$

Write down, using brackets, expressions for the questions in numbers 25-30, and simplify where possible.

25. The sum of four consecutive numbers, the smallest of which is x.

26. The sum of 3 consecutive numbers, the largest of which is n.

27. By how much is 57 greater than 19?
By how much is x greater than y?
How much is $3x + 4(a - 3b)$ greater than $8x - 5(b + 2a)$?

28. What is the perimeter of a carpet $(5x + y)$ ft long and $(2x + 3y)$ ft wide, in yd?

29. For what value of x is $3(x + 4) - 15$ equal to $2(x - 4) + 10$?

30. What is the average of the two numbers c and d? If c is itself the average of two numbers a and b, write the average without using c.

CHAPTER 9

MORE ABOUT GRAPHS

YOU HAVE already learnt how to use a formula to calculate some quantity. Now let us examine some formulae a little more closely. Take this one:

If a stone is thrown straight up with a velocity of 80 feet per second, then its height above the ground in feet is given by the formula

$$s = 80t - 16t^2$$

Where s is the height in feet and t is the time in seconds after projection.

Thus after 1 second the height is given by $s = 80 \times 1 - 16 \times 1^2$ ft
$= 80 - 16$ ft
$= 64$ ft

In the same way, after 2 seconds, $s = 80 \times 2 - 16 \times 2^2$ ft
$= 160 - 64$ ft
$= 96$ ft

We can go on to calculate the height after 3 seconds, 4 seconds, etc. To save time, the results are given in the following table:

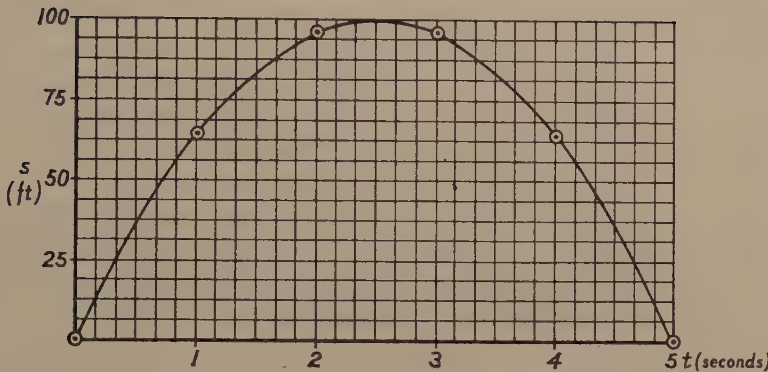
Time in seconds	0	1	2	3	4	5	6
Height in feet	0	64	96	96	64	0	− 96

The last figure will strike you as odd. Let us check it.

If t = 6, then $s = 80 \times 6 - 16 \times 6^2$
$= 480 - 16 \times 36$
$= 480 - 576$
$= -96$

So the figure is correct, but is still clearly wrong in practice: a height of − 96 feet means a depth of 96 feet below the surface of the earth. We could explain it by saying that the stone was thrown up by the edge of a cliff or a mineshaft, but otherwise we can only say that, since after 5 seconds the height of the stone is 0 feet, it has clearly hit the ground again and so the usefulness of the formula has ended. We say that the formula is *only valid for values of t between* 0 *and* 5. (It is clearly not valid for values of t less than 0, as this represents time before the stone is thrown.)

Now let us plot the points given in the table above on graph paper, and join them up with a smooth curve. The result is the figure on the opposite page.



Notice that the height axis has been placed vertically, and the time axis horizontally. Height is naturally vertical, but there is another reason. When doing the calculations from the formula, we *chose* different values of t (0, 1, 2, 3, 4, 5) and *calculated* the values of s (0, 64, 96, etc.). Both quantities vary, and so are called *variables*, but the value of s depends on what value of t we choose, so s is called the *dependent variable*, and t is called the *independent variable*.

It is agreed that when drawing a graph from a formula, the dependent variable is on the vertical axis, and the independent variable is on the horizontal axis. (Nobody knows exactly *who* agreed on this, but it is very convenient that everyone does the same thing.) The same applies when measurements are taken: if we measure the temperature at different times of day, the time is our choice, so time is the independent variable, but the temperature is measured so that is the dependent variable.

This graph can be described in two ways. We can call it a graph of the height of a stone at different times: or, much more simply, we can call it

the graph of the formula $s = 80t - 16t^2$

EXERCISE 9A

A train starts from a station. The number of feet (s) it has travelled after t seconds is given by the formula $s = \frac{1}{2}t^2$, which is valid for values of t between 0 and 9. Draw the graph of the formula $s = \frac{1}{2}t^2$ for these values of t. Use your graph to estimate (*a*) how far it has gone after 4·5 seconds; (*b*) after how many seconds it has gone 14 ft.

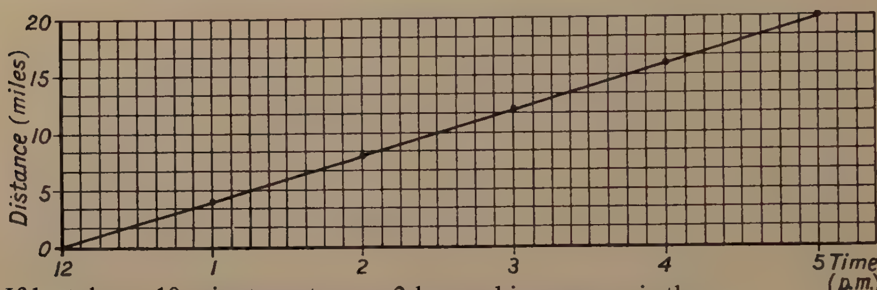
Distance-Time Graphs

The graphs we have just dealt with are called *distance-time graphs*, because they show us what distance some object has travelled in a certain time. A particularly simple type of distance-time graph occurs when a man or a vehicle travels at a constant speed.

If a man sets out at 12 noon, walking at 4 mile/h, his progress is as follows:

Time	12	1	2	3	4	5
Distance	0	4	8	12	16	20 (miles)

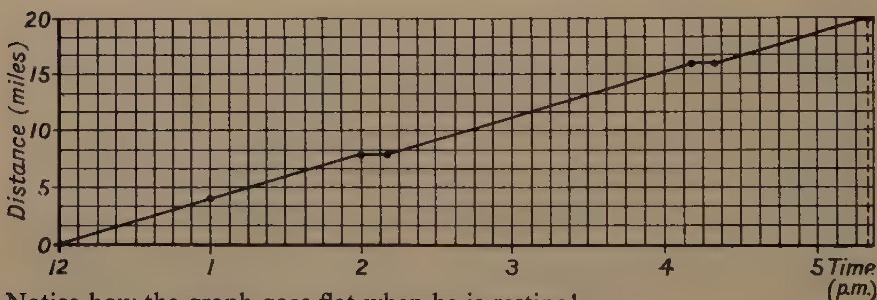
When we plot the graph we get a straight line, thus:



If he takes a 10-minute rest every 2 hours, his progress is thus:

Time	12	1	2	2.10	3.10	4.10	4.20	5.20
Distance	0	4	8	8	12	16	16	20 (miles)

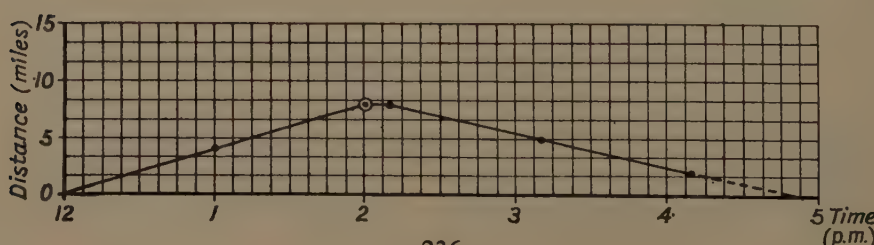
and the graph will look like this:



Notice how the graph goes flat when he is resting!

Suppose now that he walks for 2 hours, takes 10 minutes rest, and then returns at 3 m.p.h. In the table, you must remember that the distance figure represents the distance he is from his starting point (not necessarily the distance that he has walked), which is shown by the graph below:

Time	12	1	2	2.10	3.10	4.10
Distance	0	4	8	8	5	2 (miles)



We have assumed here that he does not stop to rest on the way back, but carries on after 4.10 (the dotted line) until he arrives home. At what time does the graph show that he arrived?

In fact, our table is more elaborate than it need be. The essential facts are:

(a) On the first leg of the journey he goes 8 miles in 2 hours.
(b) He rests for 10 minutes.
(c) On the return leg he will travel 6 miles back in 2 hours.

So all we want for our table is:

Time	12	2	2.10	4.10
Distance	0	8	8	2 (miles)

We can plot these four points, then join them up, producing the last line to show his arrival home. For any part of a journey travelled at constant speed, it is only necessary to plot *two* points (usually the end ones, but not always), and join them up with a straight line.

EXERCISE 9B

1. A boy sets out on a bicycle at 10 mile/h to visit a friend 24 miles away. After 2 hours he has a puncture. He stops for 20 minutes trying to mend it, but finds it impossible, so he pushes it the rest of the way at 3 mile/h. Draw a travel graph for his journey and use it to find at what time he did arrive. How much later was he than he should have been?

2. A man sets out in his car for a town 120 miles away, travelling at 30 mile/h. When he arrives he spends 20 minutes shopping and half an hour having a meal, after which he returns at 40 mile/h. Draw a travel graph for his journey, and use it to find out how long he has been away from home altogether.

A Three-Journey Graph

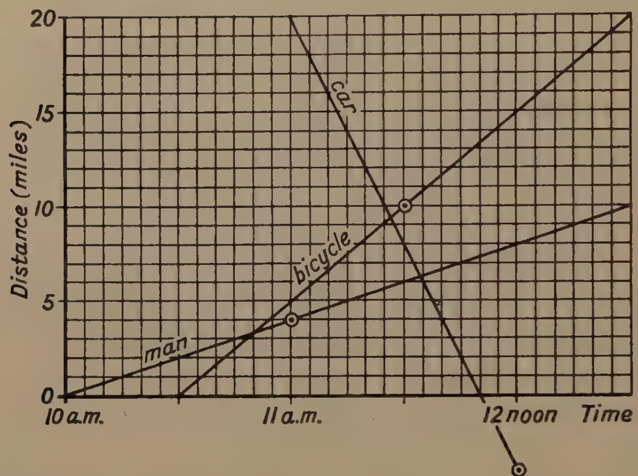
We can, if we wish, plot more than one journey on the same piece of paper.

This kind of graph is used by railway staff when planning their timetables. It helps them to see more clearly where all their trains on a particular stretch of line should be at any given moment, and so enables them to plan more efficiently to avoid accidents and delays.

Suppose we do this for the following three journeys:

(a) A man sets out walking along a road at 10 a.m., travelling at 4 mile/h.
(b) A boy on a bicycle leaves the same spot at 10.30 a.m., travelling at 10 mile/h.
(c) A car leaves a town 20 miles away at 11 a.m. and travels to meet them at 24 mile/h.

If we put the three journeys on to the same graph we get the following result:



From the graph we can see that at 11.30 a.m. the walker is 6 miles from home, the cyclist 10 miles from home, and the motorist is 8 miles from the starting point of the other two. The cyclist has already passed the pedestrian and the motorist. When and where did he do this? When and where will the motorist pass the pedestrian?

EXERCISE 9C

1. From A to B is 20 miles, and from B to C is a further 30 miles. Two ladies, one from A and one from C, decide to meet at B for a shopping spree. One lady leaves A in her car at 9 a.m. and travels to B at 30 mile/h, where she spends 2 hours shopping and then, seeing no sign of her friend, returns home again at 25 mile/h.

In fact, her friend had been delayed, and did not leave C until 11 a.m. She travelled at 40 mile/h and spent 10 minutes looking for her friend before realizing that she must have gone. She then drove on after her in the direction of A at 40 mile/h. When and where did the two ladies finally meet?

Graphs for Conversion Purposes

A straight line graph can also be useful when we have to change the unit in which some quantity is measured. Suppose that a schoolteacher has set an examination in which the maximum possible mark is 130. The headmaster requires that all results be given as percentages, i.e. as marks out of a hundred.

Thus the marks have to be altered so that any boy who got full marks should now be given 100 out of 100, any boy who got half marks (65 out of 130) should now get 50 out of 100 etc.

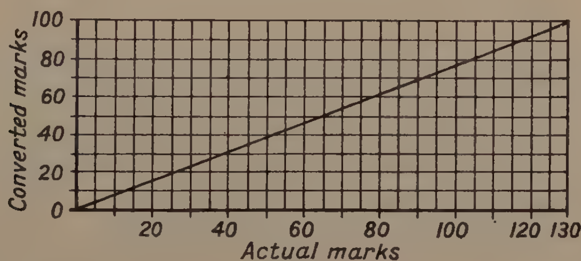
The top mark has come down from 130 to 100 $(= \frac{10}{13} \times 130)$,

so it could be done by multiplying each mark by $\frac{10}{13}$.

(For example, $\frac{10}{13} \times 65 = 50$.)

If there were a large number of marks to be altered like this, it would be a rather tedious business, but a graph will solve our problem quite easily.

We mark the point which shows 130 on the "actual mark" scale and 100 on the "converted mark" scale, and then simply join this to the point which shows 0 on both scales.

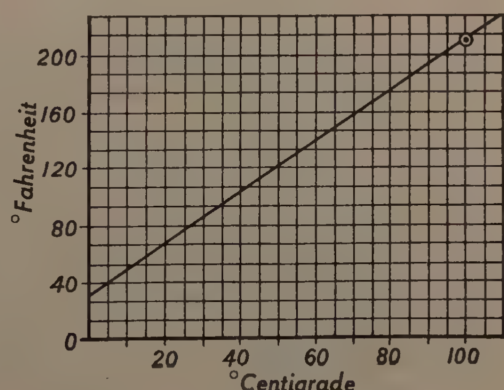


Suppose that one boy scored 78 out of 130. We find 78 on the actual mark scale, run the finger up to the line of the graph, and from there across to the converted mark scale, where we find his new mark, which is 60. What is the converted mark for a boy who actually scored 91? 52? 70? (The last one does not come to a whole number: take your answer to the nearest whole number available.)

This is called a conversion graph. Here is another one, designed to connect temperatures measured in degrees Centigrade and degrees Fahrenheit. It is based on the following facts:

Freezing point of water
0° C 32° F

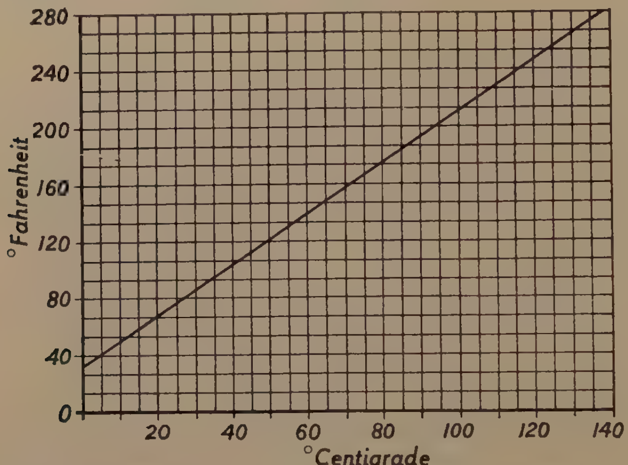
Boiling point of water
100° C 212° F



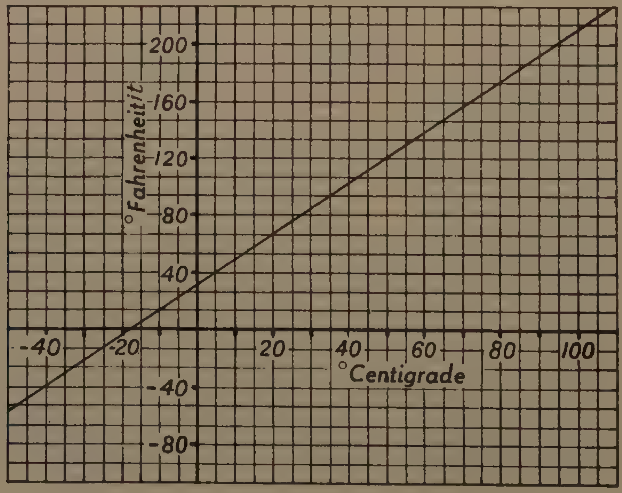
This graph can, of course, be used in both directions—to change degrees Centigrade into degrees Fahrenheit, and the other way round. Use it to find the temperature equivalent to the following:

10° C; 62° C; 47° C.   40° F; 100° F; 184° F.

To deal with such temperatures hotter than the boiling point of water we must extend the graph to the right, like this:



Now we can read off the temperature equivalent to 130° C or to 250° F. What are they? We have all known frosty weather when the temperature drops to, say 23° F. For this we must extend the graph to the left, which means that we must now draw the axes in a different place in order to leave room for temperatures which are below zero.



From this graph we see that 23° F is equivalent to − 5° C (5 degrees Centigrade below zero).

If we go even farther to the left we find that, for example, − 20° C is equivalent to − 4° F.

It is now so cold that the temperature on both scales has dropped below zero.

What temperature is equivalent to − 13° F?

There is a formula connecting these two temperature scales: it is

$$C = \frac{5}{9}(F - 32)$$

We can check it with our graph.

$$\begin{aligned} \text{When } F &= 212°, \\ C &= \frac{5}{9}(212 - 32) \\ &= \frac{5}{9} \times 180 \\ &= 100° \text{ which agrees with the graph.} \end{aligned}$$

Try checking it with (i) F = 50°; (ii) F = − 13°.

At the beginning of this chapter, we discussed the graph of the height of a stone at different times, and noted that it could also be called the graph of the formula:

$$s = 80t - 16t^2.$$

In the same way, this conversion graph can be called the graph of the formula:

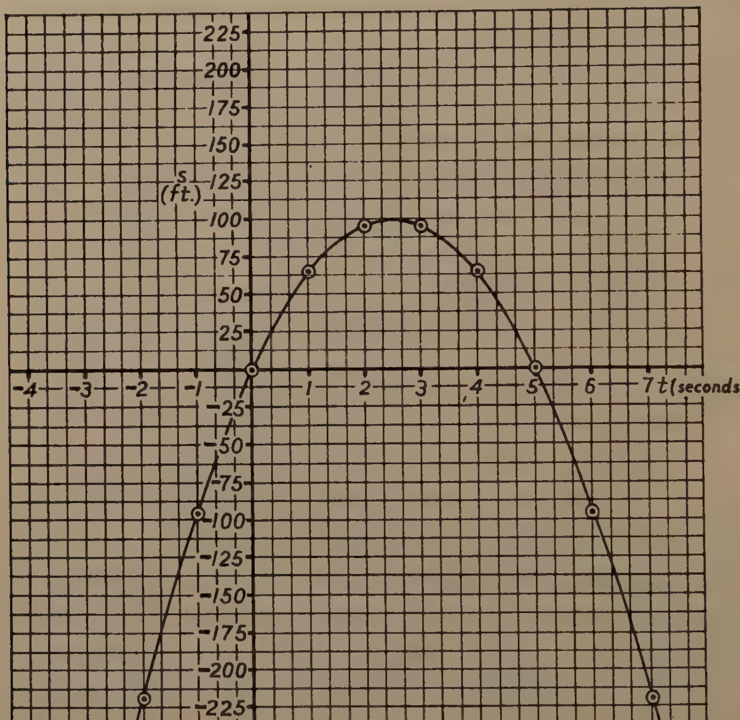
$$C = \frac{5}{9}(F - 32)$$

This temperature conversion formula works for any values we like to put in for F (or C), since minus temperatures are quite familiar, whereas the formula for the height of the stone was only valid for values of t between 0 and 5, since before $t = 0$ the stone had not been thrown, and at $t = 5$ the stone hit the ground again. But it is quite possible to extend the graph of the formula beyond $t = 5$ to the right, or $t = 0$ to the left, as long as we realize that the results have no connexion with the practical situation of a stone in the air.

We are then studying the graph of the formula $s = 80t - 16t^2$ for its own sake, without reference to the stone. Here is the original table of values extended somewhat in both directions:

t	− 2	− 1	0	1	2	3	4	5	6	7
s	− 224	− 96	0	64	96	96	64	0	− 96	− 224

and here is the graph also extended to include these extra values:



Graph of $s = 80t - 16t^2$

Without stretching the imagination too much, we can invent an illustration of this graph. Suppose that a man is in an anchored balloon 224 ft above the ground, and a stone is catapulted past him. He measures the height of the stone in relation to himself at various times, taking the instant when the stone first passes him as $t = 0$. Then at $t = 2$, say, the stone is 96 ft above him, whilst at $t = -2$ the stone is 224 ft below him, i.e. at ground level, and at $t = 6$ the stone is 96 ft below him on its way down to the ground.

In this case negative values of s indicate distances below the zero level (the balloon), and negative values of t indicate times before the zero time (when the stone passes the balloon). Nearly all graphs can have such interpretations, and hence their study is valuable even when we are not immediately concerned with the interpretation.

In order to draw the graph of a given formula, it is necessary to make a number of calculations, and this can be achieved more quickly and accurately if we set out the work in the form of a table.

Suppose we require to draw the graph of the formula:

$$y = 3x - 2$$

for values of x from -3 to $+3$. The calculations should look something like this:

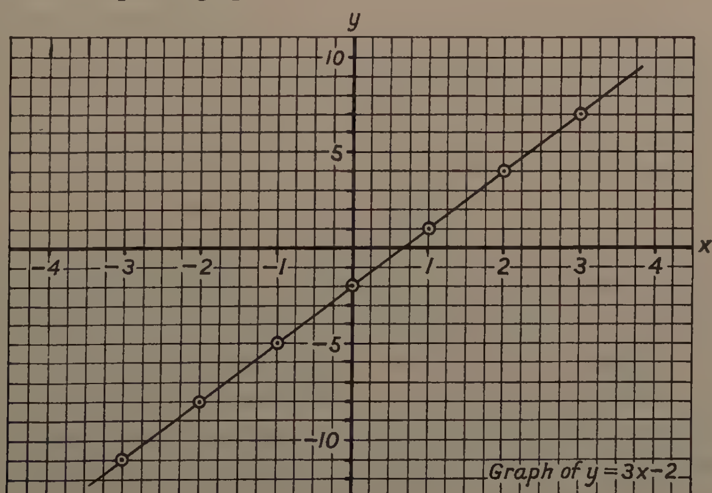
x	-3	-2	-1	0	1	2	3
$3x$	-9	-6	-3	0	3	6	9
-2	-2	-2	-2	-2	-2	-2	-2
y	-11	-8	-5	-2	1	4	7

Notice how this table is constructed. We first write down the required values of x in a row, and mark off that row with a line. The next row gives the values of $3x$, which we calculate, of course, by multiplying each value of x in turn by 3. The following row supplies the -2 required to complete the formula. (Naturally we write -2 all along the line: minus 2 is minus 2, whatever be the value of x.)

Finally we calculate each value of y by adding the two rows above, since we are told that y consists of two terms, $3x$ and -2, added together. When we come to draw the graph, we shall use only the top and the bottom rows, since we need only the values of y which go with each value of x. The two middle rows were used to help us calculate the values of y: they are not needed to draw the graph itself.

We choose scales and axes so that we can conveniently fit on to our paper values of x from -3 to 3, and values of y from -11 to 7. Then we mark the points $x = -3$, $y = -11$; $x = -2$, $y = -8$; etc.

Here is the completed graph:



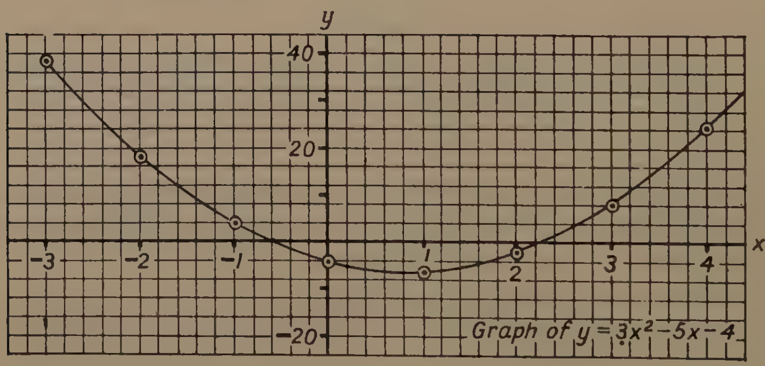
Graph of $y = 3x - 2$

Here is another example of a table of values set out in full, this time using the formula $y = 3x^2 - 5x - 4$.

x	-3	-2	-1	0	1	2	3	4
x^2	9	4	1	0	1	4	9	16
$3x^2$	27	12	3	0	3	12	27	48
$-5x$	15	10	5	0	-5	-10	-15	-20
-4	-4	-4	-4	-4	-4	-4	-4	-4
y	38	18	4	-4	-6	-2	8	24

In this more complicated example there are three terms to be added up to find each value of y, and there is a line giving the values of x^2 to help calculate the values of $3x^2$ needed for the graph. Again, once the table is complete, only the first and last rows are used for the actual plotting of the points on the graph.

Here is the completed graph:

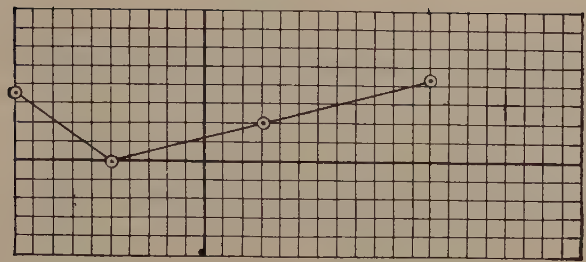


EXERCISE 9D

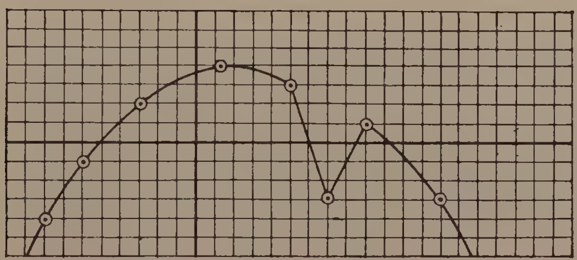
(a) $y = 4x + 1$
(b) $y = 2x^2 - 5x + 2$

Now try making out your own tables for these two formulae, using values of x from -2 to 3 in each case, and draw a graph of each result. The first should be a straight line, and the second a U-shaped curve like the one on this page.

A note on common sense. When you come to study more advanced mathematics you will meet graphs in all sorts of curious shapes; but at this stage any graph you are asked to draw will be either a straight line or a *smooth* curve with no "kinks". If you produce a graph looking like this:



or this:



then you may be sure that you have made a mistake, and should check your calculations, particularly for any odd point, as in the second diagram.

"Functions"

Throughout this chapter we have been discussing formulae which connect two letters, such as s and t, C and F, or y and x. By giving, say, x various values we were able to calculate the corresponding values of y, and as the value of x changed, so did the value of y, because the value of y depended on the value of x that we chose. The technical way we describe this relationship is to say that *y is a function of x*. This type of phrase can describe a much broader relationship than the simple formulae that we have been discussing and representing graphically. For instance, the time spent on a journey depends on how far it is and how fast you go. We know that the appropriate formula is:

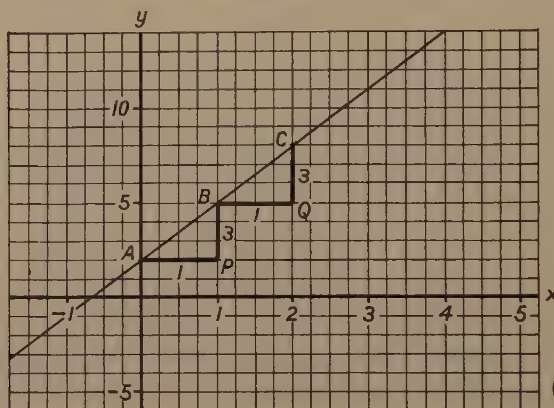
$$\text{time} = \frac{\text{distance}}{\text{speed}} \text{ or } t = \frac{d}{s}$$

so we say that time is a function of distance *and* speed, or t is a function of d and s.

This function can be expressed in a formula, as we have done here, but not all functions can be so expressed. Your height is a function of your age (among other things) but nobody can produce a formula to describe it. On the other hand, your fortune is not a function of the positions of the stars and planets when you were born, in spite of the claims of the popular newspapers.

Now we cannot usefully discuss functions which cannot be expressed in mathematical terms, so we will confine ourselves to simple formulae, and in particular let us consider those which produce a straight line when we draw their graphs.

Such functions are called LINEAR FUNCTIONS. We have already met several in this chapter: let us look at another (without for the moment bothering with the formula which it illustrates).



(Graph of $y = 3x + 2$)

The line slopes up steadily as we move from left to right. If we start at A, and take one step to the right to P, then we must take three steps up to B to rejoin the graph again. If we repeat the process, taking one step to the right from B to Q, then again we must take three steps up from Q to rejoin the graph at C; and so we can go on. Every one step to the right requires three steps up to rejoin the graph. This fact can be described in several ways:

(a) The graph is moving three times as fast in the y direction as it is in the x direction.

(b) The slope of the graph is 3 in 1.

(c) The gradient of the graph is 3.

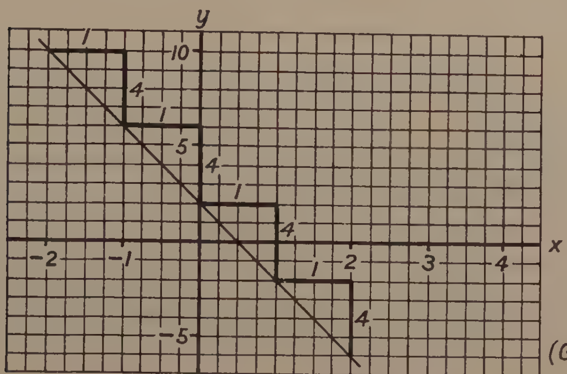
Let us look at the problem in another way. By inspecting the graph, we can find the values of y which correspond to various values of x. Here are some of them tabulated:

x	-1	0	1	2	3	4
y	-1	2	5	8	11	14

Here are the same facts emerging in another form. As x increases steadily in ones, y increases steadily in threes. Again, y is increasing three times as fast as x. Clearly the same general property will be found in any straight line graph. As x increases steadily in ones, y must also increase steadily: if y goes up in threes, we say that the gradient is 3; if y goes up in fives, we say

that the gradient is 5, and so on. (Of course, it is possible that as x goes up in ones, y goes *down* in, say fours. We then say that the gradient is -4, as in this graph):

x	-2	-1	0	1	2	3
y	10	6	2	-2	-6	-10



(Graph of $y = 2 - 4x$)

This is the characteristic property of a linear function, that as x increases steadily y also increases (or decreases) steadily. The changes in y are proportional to the changes in x. Now this will obviously happen when y is a multiple of x: suppose $y = 3x$.

Then we get a table like this:

x	1	2	3	4
y	3	6	9	12

y increases in threes; the gradient is 3. This corresponds to the 3 in the formula $y = 3x$. Now if we add, say, 2 to each value of y the y row will be altered, but *y will still increase in threes,* thus

x	1	2	3	4
y	5	8	11	14

The gradient is still 3, but since we have added 2 to each original value of y, the formula has changed from

$$y = 3x$$
$$\text{to } y = 3x + 2$$

Now there is nothing special about the numbers 3 and 2 which we have used in this example. If we had chosen 4 and 7 we would have got the formula

$$y = 4x + 7$$

and if we had chosen 6 and -3 we would have got the formula

$$y = 6x - 3$$

but the general result would have been the same: increases in y proportional to the increases in x. The graph would be a straight line and so y would be a linear function of x.

We can sum this all up by saying that if we multiply x by any number m, and then add any number c to the result, we shall have a linear function. The general formula for a linear function is

$$y = mx + c$$

The numbers m and c can be positive, negative or fractions; for example, all the following are linear functions:

$$y = 4x - 3$$
$$s = \tfrac{1}{2}t + \tfrac{3}{4}$$
$$v = -4t - \tfrac{2}{3}$$

whereas these are *not* linear functions:

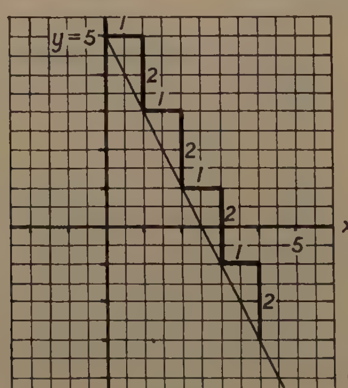
$y = 2x^2 + 5x - 7$ (containing x^2)

$p = \frac{4}{x} - 5$ (4 divided by x is not a multiple of x)

$z = \sqrt{t}$ (the square root of t is not a multiple of t)

We have also seen that the number m in the formula gives us the gradient of the graph of our linear function. In the graph of $y = 4x - 3$ the gradient is 4, that is y will go up in fours as x goes up in ones. Similarly in $s = \frac{1}{2}t + \frac{3}{4}$ the gradient is $\frac{1}{2}$, and in $v = -4t - \frac{2}{3}$ the gradient is -4 (v will go *down* in fours).

Now in the general formula $y = mx + c$, if we put $x = 0$, we find $y = c$. In other words, the number c gives the value of y at the point where the graph crosses the y axis. In the three examples given above, the graph will cross the vertical axis at $y = -3$, $s = \frac{3}{4}$, and $v = -\frac{2}{3}$ respectively.



(Graph of $y = -2x + 5$)

We are now well armed to deal with any linear function:

(a) We can recognize it because it resembles the formula $y = mx + c$.

(b) We know its graph will be a straight line.

(c) We can tell its gradient from the number m.

(d) We know where the graph will cut the vertical axis from the number c.

Let us take as an example the function $y = -2x + 5$:

It is a linear function, giving a straight line graph.

It crosses the vertical axis where $y = 5$.

Its gradient is -2, hence y will go down in twos as x goes up in ones.

Hence a *rough* idea of the graph is shown by the sketch given.

The table of values confirms our predictions:

x	0	1	2	3	4
y	5	3	1	-1	-3

Notice how the graph crosses the vertical axis (where $x = 0$) at $y = 5$, and how the values of y decrease by twos as we move along the row.

Now try to make your own *rough* diagram of the following functions, and check your work by drawing an *accurate* graph in each case to see if it resembles your first effort. To catch you out, two of them are *not* linear functions. As soon as you have spotted them, ignore them, and go on to complete the other three.

(i) $y = 3x + 1$

(ii) $s = 16t^2 + 4$

(iii) $a = \dfrac{5}{b} - 3$

(iv) $z = -3p + 8$

(v) $y = \frac{1}{2}x - 2$

CHAPTER 10

SIMULTANEOUS EQUATIONS

YOU HAVE already had a good deal of practice in solving equations such as

$$3x + 3 = 5x - 4$$

which gives the answer $x = 3\frac{1}{2}$. There is only one answer. But now look at the equation

$$x + y = 8$$

Here the problem is to find numbers for x and y such that they add up to 8. $x = 3$ and $y = 5$ will do very well, but so will $x = 2$ and $y = 6$, or $x = -5$ and $y = 13$, or $x = \frac{1}{2}$ and $y = 7\frac{1}{2}$. We could go on for ever finding pairs of values of x and y such that $x + y = 8$, i.e. the equation is satisfied. The equation has an *infinite* number of answers.

In the same way the equation

$$y - x = 4$$

has an infinite number of answers, such as $x = 3$, $y = 7$; $x = 4$, $y = 8$; $x = 2$, $y = 6$. This last pair of answers to the equation $y - x = 4$ has already appeared as a pair of answers to the equation $x + y = 8$ above.

So $x = 2$, $y = 6$ satisfies *both* the equations

$$x + y = 8$$
$$\text{and } y - x = 4$$

at the same time. We say that the solution of the *simultaneous* equations

$$x + y = 8$$
$$y - x = 4$$

is $x = 2$; $y = 6$. We found this answer by guesswork. How do we know that there isn't another solution that we have failed to guess? To show that we have in fact found the only solution, let us look at the problem from another angle.

The equation $x + y = 8$ can be rearranged as

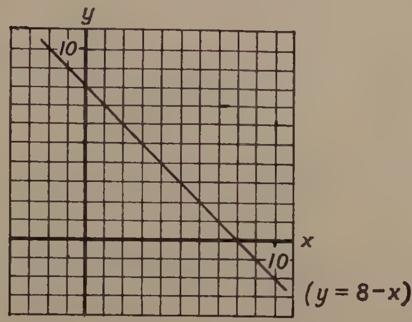
$$y = 8 - x$$

which is the same thing as

$$y = -1x + 8$$

We don't usually write $1x$ instead of just x, but we have done so here to show that y is a linear function of x: its graph will be a straight line, of gradient -1, cutting the vertical axis where $y = 8$.

It will look like this:

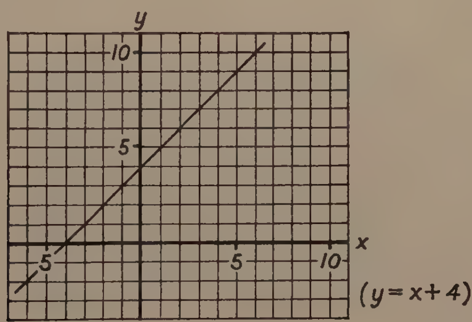


Every point on the graph will give a solution of the equation $x + y = 8$. (Check that when $x = 3$, $y = 5$ and when $x = 2$, $y = 6$.)

In the same way, the equation $y - x = 4$ can be written

$$y = 1x + 4$$

and its graph will look like this:

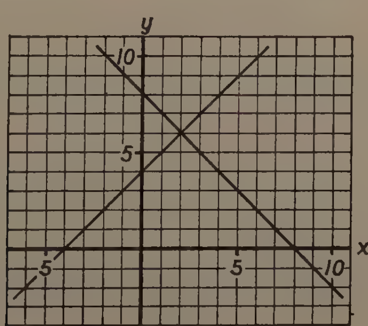


Every point on the graph will satisfy the equation

$$y - x = 4$$

(Check that when $x = 2$, $y = 6$.)

If we now draw *both* graphs on the *same* diagram, the result will look like this:



Since both graphs are straight lines, they can meet only at one point, and since that point lies on both lines, the values of x and y at that point must satisfy both equations simultaneously, in other words, the only solution of the pair of equations

$$x + y = 8 \qquad \text{and} \qquad y - x = 4$$

is given by the x and y values of the point where the two graphs intersect. (Verify that the point is in fact $x = 2$, $y = 6$.)

EXERCISE 10A

By drawing two graphs on the same diagram, solve each of the following pairs of simultaneous equations:

1. $y = 2x + 3$, $y = x + 5$ (Take values of x from -1 to 4 on each graph.)
2. $2x + y = 5$, $y - 3x = -5$ (First rewrite each equation in the form $y =$ a linear function of x; take values of x from -1 to 3.)

Equations Which Have No Solution

We said just now that two straight lines meet at a point, and hence there can be only one answer to a pair of equations like those above. But if two straight lines are parallel they will never meet, and our equations will have no solution. Now if two lines on a graph are parallel, they must have the same gradient, since if they had different slopes, they would be bound to meet somewhere, provided the graph paper were big enough. If they have the same gradient, the m part of the general formula

$$y = mx + c$$

must be the same in both equations, though the c may be different. Take two such equations, as

$$y = 2x + 1 \qquad \text{and} \qquad y = 2x + 2$$

A moment's thought tells us that these equations do, in fact, have no solution, because whatever value we give to x, the y of the second equation will always be greater than the y of the first. If $x = 1$, the values of y are 3 and 4 respectively; if $x = 2$, the values of y are 5 and 6 respectively, and so on. The fact that the graphs are parallel does not mean that the method of solution has failed; it merely confirms that the given equations have no solution.

Reference to Points or Co-ordinates

We have several times had occasion to refer to a particular point on a graph, such as the point where $x = 2$ and $y = 6$. This is a rather long and clumsy way of putting it, and so mathematicians have invented a shorter method. They simply refer to the point (2, 6). In the same way, the point

where $x = 5$ and $y = \frac{1}{2}$ is called the point $(5, \frac{1}{2})$. The value of x is written first, and the value of y second; what could be simpler? Let us refer back to the pair of equations at the beginning of the chapter:

$$x + y = 8$$
$$y - x = 4$$

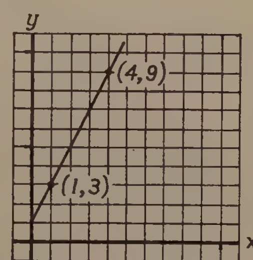
We can say either that the solution of these equations is

$$x = 2, y = 6$$

or we can look at it from a graphical point of view and say that the *lines* $x + y = 8$ and $y - x = 4$ *intersect at the point* (2, 6).

When we use this shorthand notation to refer to a point such as $(-1, 4)$, the number -1 is called the x *co-ordinate of the point*, and the number 4 is called the y *co-ordinate*. The pair of numbers $(-1, 4)$ together are simply called the *co-ordinates*.

Now suppose we take a piece of graph paper and mark on it the two points whose co-ordinates are (1, 3) and (4, 9), and then join them up with a straight line, like this:



The problem we set ourselves is to find the equation of the line (that is, the formula which will give us this graph). To begin with, we know that the graph is straight, so y is a linear function of x. Hence the equation will look like

$$y = mx + c$$

where the values of m and c are the numbers we seek. Now the point (1, 3) lies on this line, and so the values $x = 1$, $y = 3$ must satisfy the equation $y = mx + c$. By substituting these values of x and y, we find that

$$3 = m + c$$

By using the same argument about the point (4, 9) we find that

$$9 = 4m + c$$

Now we have two equations

$$3 = m + c$$
$$9 = 4m + c$$

and we have to find values of m and c which satisfy both of them at once; in other words, we have to solve a pair of simultaneous equations. In this case, it is not difficult to guess the answers, which are:

$$m = 2, c = 1$$

Now that we have found m and c, we can substitute these values in the general equation $y = mx + c$, and so find that

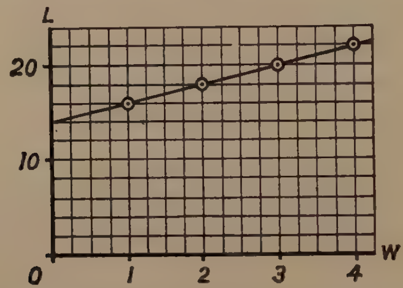
$$y = 2x + 1$$

This is the equation of the line that we sought. (Check that the points (1, 3) and (4, 9) do in fact lie on it; when $x = 1$, does $y = 3$?)

All this may seem rather academic, but in fact it often happens that a scientific experiment will give a number of measurements which, when plotted on a graph, give a straight line. A simple example is given by the measurements of the length of a spring when various weights are hung on it. The following table gives the sort of results obtained:

Length in inches (L)	14	16	18	20	22
Weight in pounds (W)	0	1	2	3	4

The graph, drawn with L vertical and W horizontal, looks like this:



Clearly L is a linear function of W, and so the formula connecting them must look like

$$L = mW + c$$

Also, the graph passes through five points whose co-ordinates we know; we only need two of them. Let us choose the points (0, 14) and (4, 22). Since $L = 14$ when $W = 0$ we have

$$14 = 0\,m + c$$

$$\text{and similarly } 22 = 4\,m + c$$

A pair of simultaneous equations again, but very simple indeed this time: as $0m = 0$, the first equation tells us that $14 = c$, and if we now put $c = 14$ in the second equation, we have

$$22 = 4\,m + 14$$

$$8 = 4\,m$$

$$2 = m$$

Hence the formula connecting L and W for this spring is

$$L = 2W + 14$$

In fact, our experimenter would probably argue something like this:

L goes up in twos as W goes up in ones, hence the gradient of the line is 2; that is, $m = 2$. Also the graph cuts the vertical axis where $L = 14$, therefore $c = 14$.

This is a much quicker argument when the graph has already been drawn, but less convenient when we have to start at the beginning with only the figures.

EXERCISE 10B

1. Find the equation of the line joining the points (0, 4) and (2, 10).

2. The speed of a car was measured at intervals along a short stretch of road and the results tabulated as follows:

Speed in feet per second (v)	12	22	32	42	52
Time after zero in seconds (t)	0	5	10	15	20

Draw a graph to illustrate these figures, taking the time axis horizontal. Find an equation of the form $v = mt + c$ connecting v and t. (The value of m in this example is called the *acceleration* of the car.)

Methods of Solving Equations

The pairs of simultaneous equations which have appeared so far in this chapter have been solved mostly by guesswork. Clearly we want a more reliable way to cope with harder problems. There are two main methods, which are best illustrated by examples.

(a) *The substitution method.*

Examples:

Consider the equations

$$3x + y = 22 \qquad (1)$$
$$y = x + 6 \qquad (2)$$

Notice how the equations are labelled (1) and (2). This helps us to explain clearly the steps of our solution. Now equation (2) tells us that y is equivalent to $(x + 6)$, so we may replace the letter y by the expression $(x + 6)$ in equation (1), giving a third equation

$$3x + (x + 6) = 22 \qquad (3)$$

This process is called substituting equation (2) in equation (1).

Now equation (3) is a quite straightforward type we have met before. We have

$$3x + x + 6 = 22$$
$$3x + x = 22 - 6$$
$$4x = 16$$
$$x = 4$$

Next we put $x = 4$ in equation (2), getting

$$y = 4 + 6$$
$$y = 10$$

Finally, we check our answer in the remaining equation, (1). The left-hand side is

$$\begin{aligned} 3x + y &= 12 + 10 \text{ (since } x = 4 \text{ and } y = 10) \\ &= 22 \\ &= \text{R.H.S. (Right-hand side)} \end{aligned}$$

which confirms our solution.

Here is another one, written out in the correct manner and using accepted abbreviations:

Example:

Solve the equations:

$x = 3 - 2y$ (1) and $4x - 2y = -28$ (2)

Subst. (1) in (2):

$$4(3 - 2y) - 2y = -28$$
$$12 - 8y - 2y = -28$$
$$-8y - 2y = -28 - 12$$
$$-10y = -40$$
$$10y = 40,\ y = 4$$

Subst. in (1):

$$x = 3 - 8$$
$$x = -5$$

Check in (2):

$$\text{L.H.S.} = 4x - 2y$$
$$= -20 - 8$$
$$= -28$$
$$= \text{R.H.S.}$$

Answer: $x = -5,\ y = 4.$

Notice the steps of the solution:

(a) If one equation reads $x =$ something or $y =$ something, take that equation and substitute it in the other.
(b) Solve the new equation you have made.
(c) Take your answer and substitute it in the *easier* of the two original equations.
(d) Check both answers in the remaining equation.
(e) Write your answers clearly at the end.

EXERCISE 10C

Solve the following pairs of equations:

1. $x = y + 3$
$3y = x - 1$

2. $2x + 4y = 10$
$y = 3x - 8$

3. $y = 2x - 7$
$x - 3y = 11$

4. $y = 4x - 1$
$x + 2y = 2\frac{1}{2}$ (one answer is a fraction)

5. $3x - 4y = -9$
$x = 7y - 2$

Rearrangement of Equations

The method of substitution is very convenient when one equation reads $x =$ something or $y =$ something, but unfortunately we often find that neither equation is of this type. Sometimes we can get round the difficulty with a little ingenuity, as in the following examples.

(a) $3x + y = 18$ (1) $\quad 2x - 3y = 1$ (2)

Rearrange equation (1) in the form $\quad y = 18 - 3x$ (3)

Now substitute (3) in (1) $\quad 2x - 3(18 - 3x) = 1$

and we can continue as before. Try to finish this one off yourself.

(b) $2x - 7y = -1$ (2) $\quad 5y = 4x - 7$ (2)

Rearrange equation (1) in the form $\quad 2x = 7y - 1$ (3)

Now multiply equation (3) by 2 $\quad 4x = 14y - 2$ (4)

Now substitute equation (4) in equation (2)

$$5y = (14y - 2) - 7$$
$$5y - 14y = -2 - 7$$
$$-9y = -9$$
$$9y = 9$$
$$y = 1$$

Substitute in (1)

$$2x - 7 = -1$$
$$2x = -1 + 7$$
$$2x = 6$$
$$x = 3$$

Check in (2) L.H.S. $= 5y$ $\quad$ R.H.S. $= 4x - 7$

$= 5$ $\quad = 12 - 7 = 5$

Answer: $x = 3$, $y = 1$.

EXERCISE 10D

Solve the following pairs of equations.

1. $x + 2y = 19$
   $3x - 5y = -20$

2. $3x + 7y = 20$
   $14y - 5x = 18$

The Elimination Method

As the equations get more complicated, the method of substitution leads to very heavy manipulation. In such circumstances, it is better to use the second method, known as the *method of elimination*.

Example:

$$7x + 5y = 11 \qquad (1)$$
$$3x - 5y = 19 \qquad (2)$$

If we add the two equations together, we have

$$(7x + 3x) + (5y - 5y) = 11 + 19$$
$$\text{or } 10x = 30 \quad (3)$$
$$\text{giving } x = 3$$

and by substitution in (1)

$$21 + 5y = 11$$
$$5y = 11 - 21$$
$$5y = -10$$
$$y = -2$$

The solution was possible because the terms $(+ 5y)$ and $(- 5y)$ add up to zero, and so no term involving y appeared in equation (3). In other words, the letter y was eliminated from the work, giving us a simple equation in x only. It is clearly necessary to have the terms in y balanced (equal in value) but opposite in sign. In the next example it is the terms in x which are balanced, but they are *not* opposite in sign:

$$3x - 4y = - 11 \quad (1) \qquad\qquad 3x + 2y = 1 \quad (2)$$

To make the x terms opposite in sign, we take one of the equations and change the sign of *every* term in it. This leaves the equation essentially unaltered. Either equation will do: suppose we choose (1). We show what we have done by writing

(1) × (− 1). (This means Equation (1) multiplied by minus 1.)

Equation (2) is left unaltered, and we now have this situation:

$$\begin{array}{lr} (1)\times(-1) & -3x + 4y = 11 \\ (2) & 3x + 2y = 1 \\ \text{adding as before,} & 6y = 12 \\ & y = 2 \end{array}$$

and we complete the solution by substituting $y = 2$ in (1), finding $x = - 1$, and checking in (2).

EXERCISE 10E

Solve the following equations by the method of elimination.

1. $6x + y = 8$, $5x - y = 14$ **2.** $4x - 3y = 8$, $4x + 2y = - 2$ **3.** $5x - 3y = 6$, $2x - 3y = - 3$

This method deals very satisfactorily with the problem of finding the equation of the straight line graph joining two given points.

Example:

Find the equation of the line joining the points $(2, - 6)$ and $(4, 7)$.

Solution: The equation will be of the form

$$y = mx + c$$

Since the point $(2, - 6)$ lies on the line, when $x = 2$, $y = - 6$.

$$\begin{array}{lrl} \text{Hence} & -6 = 2m + c & (1) \\ \text{Similarly,} & 7 = 4m + c & (2) \\ (1) \times -1 & 6 = -2m - c & \\ (2) & 7 = 4m + c & \\ \text{adding,} & 13 = 2m & \\ & m = 6\tfrac{1}{2} & \\ \text{Subst. in (1),} & -6 = 13 + c & \\ & -6 - 13 = c & \\ & c = -19 & \end{array}$$

Check in (2), $\quad$ R.H.S. $= 4m + c$
$$= 26 - 19$$
$$= 7$$
$$= \text{L.H.S.}$$

The equation of the line is therefore
$$y = 6\tfrac{1}{2}x - 19$$

EXERCISE 10F

Find the equations of the lines joining the following pairs of points:

1. (3, 2) and (4, 6) **2.** (− 1, 5) and (3, 9) **3.** (− 2, 4) and (1, − 2)

Balancing the Letters of an Equation

We have seen that the essential point of the method of elimination is to get the terms in one of the letters balanced and of opposite sign. If they are not, we can always make them so, as in the following cases:

$$3x - y = 1 \qquad (1)$$
$$5x + 2y = 20 \qquad (2)$$

If we multiply (1) by 2, and leave (2) as it is, we get

(1) $\times$ 2 $\qquad 6x - 2y = 2$
(2) $\qquad 5x + 2y = 20$

and the terms in y are ready for elimination.

Example:

$$4x + 3y = 1 \qquad (1)$$
$$5x + 4y = 2 \qquad (2)$$

If we multiply (1) by 5, *and also* (2) by − 4, we get

(1) $\times$ 5 $\qquad 20x + 15y = 5$
(2) $\times - 4$ $\qquad -20x - 16y = -8$

and now the terms in x are ready for elimination. Notice that if we had used 5 and 4 as multipliers, we should have balanced the terms in x, but with the *same* sign. By using − 4 instead of 4, we obtained the required *opposite* signs as well.

EXERCISE 10G

1 and **2.** Complete the solutions of the two examples above.

Now solve these equations; in some of them a letter after the question in brackets, thus (x), is a hint to start by eliminating that letter.

3. $5x - y = 3$, $2x + 3y = 8$ (y)

4. $2x + 5y = 9$, $4x - 3y = -8$ (x)

5. $3x + 4y = 5$, $2x - 3y = 9$ (y)

6. $x + 2y = 11$, $2x - y = 2$

7. $2x + 5y = 8$, $3x + 4y = 5$

Non-standard Equations

In all the equations discussed so far, the general layout has been the same:

(so many) x and (so many) y = a number

If the equations are not given in this standard form, then our first task must be to rearrange them until they *are* in this form. Here are some more equations.

Examples:

(a) $2a = 3b - 7$ (1) $\qquad$ $4b + 5a = 40$ (2)

We can rearrange them as follows:

(1) $2a - 3b = -7$ $\qquad$ (2) $5a + 4b = 40$

In (1) we transferred the term $3b$ to the left-hand side of the equation. In (2) we merely altered the order of the terms on the left, so as to get the corresponding letters in the same position.

(b) $m = 3s$ (1) $\qquad$ $m + 12 = 2(s + 12)$ (2)

(1) can be written $\qquad m - 3s = 0 \qquad$ (3)

(2) takes longer. First we clear the bracket.

$$m + 12 = 2s + 24$$

rearranging, $m - 2s = 24 - 12$

$$m - 2s = 12 \qquad (4)$$

Now (3) and (4) are a pair of equations in standard form:

$m - 3s = 0$ (3) $\qquad$ $m - 2s = 12$ (4)

(c) $\dfrac{x}{4} - \dfrac{y-3}{2} = 1$ (1) $\qquad$ $\dfrac{y+1}{3} = \dfrac{1}{5}(x + 2)$ (2)

(1) × 4 gives

$$x - 2(y - 3) = 4$$
$$x - 2y + 6 = 4$$
$$x - 2y = 4 - 6$$
$$x - 2y = -2 \qquad (3)$$

(2) × 15 gives

$$5(y + 1) = 3(x + 2)$$
$$5y + 5 = 3x + 6$$
$$-3x + 5y = 6 - 5$$
$$-3x + 5y = 1 \qquad (4)$$

(3) and (4) are in standard form:

$$x - 2y = -2 \qquad (3)$$
$$-3x + 5y = 1 \qquad (4)$$

You should verify that the solution of these equations is $x = 8, y = 5$.

When we come to check our answer, it is important to choose one of the *original* equations, such as (1), in case a mistake has been made in deriving equations (3) and (4).

In (1), then,

$$\begin{aligned} \text{L.H.S.} &= \frac{x}{4} - \frac{y-3}{2} \\ &= \frac{8}{4} - \frac{5-3}{2} \\ &= 2 - 1 \\ &= 1 = \text{R.H.S.} \end{aligned}$$

EXERCISE 10H

Solve the following:

1. $3b = 2a + 3$
   $5a - 7b = 2$
2. $f + 4 = 3(s + 4)$
   $f - 4 = 5(s - 4)$
3. $\frac{p}{6} + 4 = \frac{q}{2}$
   $\frac{p-3}{7} - \frac{q-2}{3} = 2\frac{1}{3}$

Procedure for Making Equations

Simultaneous equations arise from all sorts of problems in which two quantities are unknown and have to be found. It is quite impossible to give a complete classification of the various types; we must simply treat each problem on its merits. Read the question carefully, and choose letters to represent the unknown quantities which have to be found. *Begin the solution by stating clearly what each letter represents.*

The question will usually divide naturally into two halves; take each half in turn and translate it into a mathematical equation. You will then have two equations, which you can solve by whatever method you think most suitable. The examples which follow are worked out in full to emphasize the correct method of writing out solutions.

(a) A cricket bat and two sets of stumps cost £3·15 altogether. The bat alone costs £1·95 more than one set of stumps. Find the price of a bat and of a set of stumps.

Solution:

Let the price of a bat be b pence.

Let the price of a set of stumps be s pence.

Then $b + 2s = 315\text{p}$ (1) $\quad b = s + 195\text{p}$ (2)

Subst. (2) in (1),

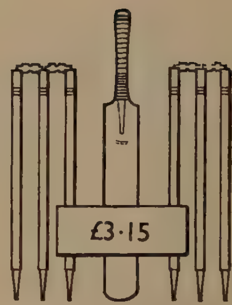
$$\begin{aligned} s + 195 + 2s &= 315 \\ s + 2s &= 315 - 195 \\ 3s &= 120 \\ s &= 40 \end{aligned}$$

Subst. in (2),

$$\begin{aligned} b &= 40 + 195 \\ b &= 235\text{p} = £2{\cdot}35 \end{aligned}$$

Check: 1 bat £2·35
2 sets stumps £0·80
TOTAL £3·15

Answer: A bat costs £2·35
A set of stumps cost 40p



Note these points:

(1) The first two sentences of the problem each give us one equation.

(2) As we are working in pence, the £3·15 must be changed to pence.

(3) For our check we return to the original problem, in case a mistake was made in writing down the equations.

(b) A man sets out on a journey of 72 miles in his car. He travels part of the way at 24 mile/h and the rest of the way at 30 mile/h. The whole trip takes $2\frac{1}{2}$ hours. How far did he travel at each speed?

Solution:

Let the distance travelled at 24 mile/h be x miles.
Let the distance travelled at 30 mile/h be y miles.

x miles at 24 mile/h takes $\dfrac{x}{24}$ hours.

y miles at 30 mile/h takes $\dfrac{y}{30}$ hours.

The total time is given as $2\frac{1}{2}$ hours.

$$\frac{x}{24} + \frac{y}{30} = 2\tfrac{1}{2} \qquad (1)$$

The total distance is given as 72 miles.

$$x + y = 72 \qquad (2)$$

(1) × 120, $\quad 5x + 4y = 300 \qquad (3)$

(2) × − 4, $\quad -4x - 4y = -288 \qquad (4)$

Adding, $\quad x = 12$

Subst. in (2), $\quad 12 + y = 72$

$\quad y = 60$

Check: 12 miles at 24 mile/h takes $\frac{1}{2}$ hour
60 miles at 30 mile/h takes 2 hours

Total time $2\frac{1}{2}$ hours

Answer: 12 miles at 24 mile/h. 60 miles at 30 mile/h.

Note how equation (1) was formed by writing down an expression for the *time* spent on each part of the journey and then using the given fact that the total time was $2\frac{1}{2}$ hours. This device is very common in travel problems.

(c) Four years ago, a father was five times as old as his son. In four years' time, he will be only three times as old as his son. Find their present ages.

Solution:

Let the father's present age be f years.
Let the son's present age be s years.
In four years' time, the father's age will be $f + 4$ years,
the son's age will be $s + 4$ years.

$$f + 4 = 3(s + 4) \qquad (1)$$

similarly,
$$f - 4 = 5(s - 4) \qquad (2)$$

Simplifying (1),
$$f + 4 = 3s + 12$$
$$f - 3s = 8 \qquad (3)$$

Simplifying (2),
$$f - 4 = 5s - 20$$
$$f - 5s = -16 \qquad (4)$$

(3)
$$f - 3s = 8$$
(4) $\times -1$,
$$-f + 5s = 16$$
adding,
$$2s = 24$$
$$s = 12$$
Subst. in (3),
$$f - 36 = 8$$
$$f = 44$$

Check: 4 years ago, son 8, father 40. Five times as old.
4 years time, son 16, father 48. Three times as old.

Answer: The father is 44, his son is 12.

(d) A handful of coins, five in number, consists of 5p and 10p pieces. The coins are worth 35p altogether. How many are there of each?

Solution:

Let the number of 5p coins be x.
Let the number of 10p coins be y.
The *value* of x 5p coins is $5x$p.
The *value* of y 10p coins is $10y$p.

So $$5x + 10y = 35 \qquad (1)$$

Since there are five coins,
$$x + y = 5 \qquad (2)$$

(1) $\div$ 5
$$x + 2y = 7$$
(2) $\times (-1)$
$$-x - y = -5$$
Adding,
$$y = 2$$
Subst. in (2),
$$x = 3$$

Check:

$$3 \times 5\text{p} = 15\text{p}$$
$$2 \times 10\text{p} = 20\text{p}$$
$$\text{TOTAL} \quad 35\text{p}$$

Answer: 3 coins worth 5p, 2 coins worth 10p

Note how the two equations were formed.
(2) arose from the *number of coins.*

(1) arose from the *total value* of the coins. To find this, we had to express the value of the coins in some common unit (here pence) and equate it to the given total value (35p) *also in the same unit.*

(e) A number consists of two digits whose sum is 10. If the number is subtracted from the number formed by reversing the original digits, the result is 18. Find the number.

Solution:

Let the tens digit be x.
Let the units digit be y.
Then the number consists of x tens and y units, i.e. $10x + y$. (Just as 36 consists of $30 + 6$.)
The number formed by reversing the digits is $10y + x$.
The difference between these is given as 18.

$$10y + x - (10x + y) = 18 \qquad (1)$$

Since the sum of the digits is 10,

$$x + y = 10 \qquad (2)$$

Simplifying (1), $10y + x - 10x - y = 18$

$$9y - 9x = 18$$
$$y - x = 2 \qquad (3)$$

Rearrange (2), $y + x = 10$

adding, $2y = 12$

$$y = 6$$

Subst. in (2), $x = 4$

Check: $64 - 46 = 18$

Answer: The number is 46.

EXERCISE 10J

This exercise contains a number of problems, many of them similar to the preceding examples, but not all of them. Remember that the golden rule for solving problems is to *think carefully*. There is no rule to be learned by heart.

1. Two locomotives and three coaches for a model railway cost £7·85. One locomotive and six coaches would cost £5·95. Find the cost of a locomotive and of a coach.

2. A man is three times as old as his son. In 11 years' time, he will be only twice as old as his son. Find their present ages.

3. A boy cycles to the nearest town by the main road at 12 mile/h and returns by the side roads at 10 mile/h. The return journey is two miles longer than the outward journey, and it takes 15 minutes more. How long is each route? (Remember to convert minutes into hours.)

4. A pile of 2p pieces and 5p pieces is worth 43p. There are four more 2p pieces than 5p pieces. How many coins are there of each denomination?

5. A two-digit number and the number formed by reversing its digits are added together; the result is 121. The tens' digit is 3 less than the units' digit. Find the number.

6. If John gives Mary 10p, he will have twice as much money as she has. If Mary gives John 10p, he will have three times as much as she has. How much money have they each?

7. A boy sets out on his bicycle for a town 26 miles away, cycling at 10 mile/h. Near his destination he has a puncture, fails to mend it after 15 minutes, and so walks the rest of the way at 4 mile/h. The whole journey takes him 3 hours. How long did he spend cycling, and how long walking?

CHAPTER 11

SIMPLE FACTORS, SQUARES AND PRODUCTS

THE PROCESS of finding factors is one of the most important in algebra, and so it is a good idea to be quite clear as to what is meant by the word factor, and also what it does *not* mean. Look at these examples:

$7 \times 8 = 56$ .. .. 7 and 8 are factors of 56
$14 \times 4 = 56$ .. .. 14 and 4 are factors of 56
$28 \times 2 = 56$ .. .. 28 and 2 are factors of 56
$41 + 15 = 56$ .. .. 41 and 15 are *not* factors of 56

Any numbers which, *when multiplied together*, make 56, are *factors* of 56.

$a \times b^2 = ab^2$ .. .. a and b^2 are factors of ab^2
$ab \times b = ab^2$ .. .. ab and b are factors of ab^2
$2(t^2 + 3t) = 2t^2 + 6t$ .. 2 and $(t^2 + 3t)$ are factors of $2t^2 + 3t$
$t(2t + 6) = 2t^2 + 6t$ .. t and $(2t + 6)$ are factors of $2t^2 + 6t$
$2t(t + 3) = 2t^2 + 6t$ .. $2t$ and $(t + 3)$ are factors of $2t^2 + 6t$

Any letters, or expressions containing letters, which, *when multiplied together*, make ab^2 or $2t^2 + 6t$, are *factors* of ab^2 or $2t^2 + 6t$. Notice that the expressions containing letters may be either combinations of letters or letters and numbers, such as ab, b^2, $2t$, or they may be complete brackets, such as $(t + 3)$. The process of finding such factors is called *factorization*.

To sum up, any number, or letter, or algebraic expression, which will divide into another number or expression without remainder, is called a *factor* of the number or expression. It may be easy to find one factor, but difficult or impossible to find others. The number 2 is clearly a factor of 6,483,254,638, but to find all of them would take a long time. The letter x is a factor of $x^3 - 3x^2 + 2x$, since it will divide into the expression exactly. Later in this chapter you will learn how to find the other factors.

As can be seen in the above examples, there are often several ways in which an expression can be factorized. We have given three possible factorizations of 56. Here are two more:

$$7 \times 4 \times 2 = 56$$
$$7 \times 2 \times 2 \times 2 = 56$$

This last statement you will probably recognize as giving the *prime* factors of 56. It is impossible to split 56 into *more* than four factors. Another way of putting this is to say that 56 is *completely factorized* as $7 \times 2 \times 2 \times 2$, or

7×2^3. In the same way, we have given three ways of factorizing $2t^2 + 6t$. In the first two cases, we have split it into two factors ($2 \times$ a bracket and $t \times$ a bracket), but in the last case we have split it into *three* factors ($2 \times t \times$ a bracket). It is impossible to split it into more than three factors, so we say that $2t^2 + 6t$ is completely factorized as $2t\,(t + 3)$.

The method of finding such factors is very simple. Take the expression

$$4ab - 6b^2$$

First we examine it to see if there is any number or letter which divides into *both* terms of the expression. We see that the number 2 and the letter b appear in the two terms $4ab$ and $6b^2$, and so are factors of the expression. We take these out in front, and then work out the bracket that will be left by common sense. The factors of $4ab - 6b^2$, then, are

$$2b\,(2a - 3b)$$

A mental multiplication of this expression gives us $4ab - 6b^2$, which serves as a check on our work. This should always be done.

Here are some more examples to show the sort of answers which we get from factorization questions:

$$6c + 6d = 6\,(c + d)$$
$$5a^2 - ab = a\,(5a - b)$$
$$2x + 4 = 2\,(x + 2)$$
$$c^2 + c = c\,(c + 1)$$ (Note this type; it often gives trouble.)
$$4xy^2 + 2y^2 = 2y^2\,(2x + 1)$$
$$3x + 3y + 6z - 9t = 3\,(x + y + 2z - 3t)$$

EXERCISE 11A

Factorize the following expressions:

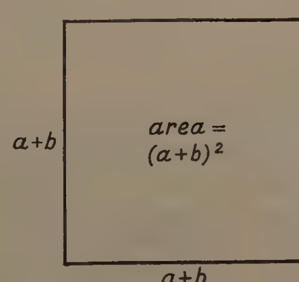
1. $2x - 4y$
2. $3ab + 9bc$
3. $p^2 - pq$
4. $x^3 - x^2y$
5. $3b + 3$
6. $d^2 - d$
7. $3h^2 + h$
8. $2rs - s$
9. $t + 4t^3$
10. $5a^2b - 10ab^2$
11. $4c^3 + 8c$
12. $4xy - 2xz + 8x^2$

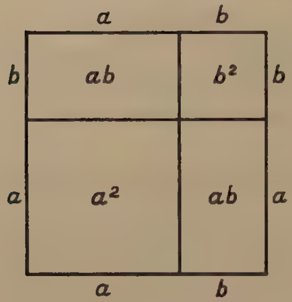
The Squaring of Bracketed Expressions

You may remember that the reason for calling x^2 "x-squared" is that we can represent x^2 by the area of a square of side x units. In the same way we can represent the expression

$$(a + b)^2$$

by the area of a square of side $(a + b)$ units, thus:





Let us analyse this area a little more closely, dividing it up into sections as in the accompanying diagram:

We now see that the area consists of four parts: a square of area a^2, a square of area b^2 and two rectangles of area ab. The total area is

$$a^2 + b^2 + 2ab$$

In other words,

$$(a + b)^2 = a^2 + b^2 + 2ab$$

Notice very carefully the shape of the expression on the right, because this is one of the most important formulae in algebra. Beginners often make the mistake of thinking that to square a bracket such as $(a + b)$ it is sufficient to square the two terms in the bracket. This is not true. There is a third term to be put in, the term $2ab$: i.e., twice the product of a and b. A little simple arithmetic will soon verify this, if the diagram above is not convincing:

$$(4 + 6)^2 = 10^2 = 100$$
$$4^2 + 6^2 = 16 + 36 = 52$$

The difference between 100 and 52 is 48, and the missing 48 is supplied by the third term in the formula, i.e. twice the product of 4 and 6, which is

$$2 \times 4 \times 6 = 48$$

Check again with some different numbers:

$$(5 + 2)^2 = 7^2 = 49$$

and, from the formula, $(5 + 2)^2 = 5^2 + 2^2 + (2 \times 5 \times 2)$

$$= 25 + 4 + 20$$
$$= 49$$

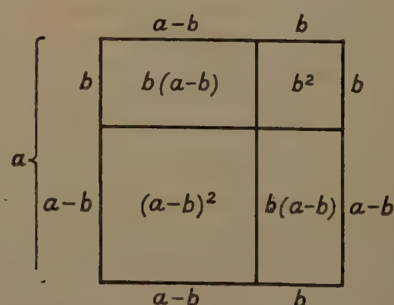
Check for yourself with the following:

(i) $(10 + 2)^2$;
(ii) $(12 + 8)^2$.

Naturally there is a similar formula to calculate:

$$(a - b)^2$$

But the diagram with which we illustrate it is a little more complicated. Study it carefully.



We get the area $(a - b)^2$ by taking the large square a^2 and subtracting from it the small square b^2 and two rectangles of area $b\,(a - b)$, that is,

$$(a - b)^2 = a^2 - b^2 - 2b\,(a - b)$$
$$= a^2 - b^2 - 2ab + 2b^2$$
$$= a^2 + b^2 - 2ab$$

Compare this formula with the last one:

$$(a + b)^2 = a^2 + b^2 + 2ab$$
$$(a - b)^2 = a^2 + b^2 - 2ab$$

We see that they differ only in the sign of the $2ab$ term, which is in each case the same as the sign in the original bracket. Note particularly that both the a^2 and the b^2 are *positive.*

We have already checked the first formula by substituting various numbers in place of a and b: let us check the second by putting $a = 6$, $b = 5$.

$$(6 - 5)^2 = 1^2 = 1$$

and by the formula, $(6 - 5)^2 = 6^2 + 5^2 - 2 \times 6 \times 5$

$$= 36 + 25 - 60 = 1$$

The chief advantage of these formulae, however, is that we can replace a and b, not just by numbers, but by complete algebraic expressions. If in the formula for $(a + b)^2$ we put $a = 4x$ and $b = 3y$, we have:

$$(4x + 3y)^2 = (4x)^2 + (3y)^2 + 2\,(4x)\,(3y)$$
$$= 16x^2 + 9y^2 + 24xy$$

Similarly, replacing a by $3a$ and b by 5 in the formula for $(a - b)^2$,

$$(3a - 5)^2 = (3a)^2 + 5^2 - 2\,(3a)\,(5)$$
$$= 9a^2 + 25 - 30a$$

Thus we can square any bracket containing two terms (such an expression is called a *binomial*) by using one of the two formulas above.

EXERCISE 11B

Work out the following squares:

1. $(p + q)^2$
2. $(x - y)^2$
3. $(2a + b)^2$
4. $(2c - 5)^2$
5. $(6x + 1)^2$
6. $(4y - 7z)^2$
7. $(h^2 + k^2)^2$

Finding the Difference Between Squared Numbers

Look at this short column of calculations:

$$3^2 - 2^2 = 9 - 4 = 5$$
$$4^2 - 3^2 = 16 - 9 = 7$$
$$5^2 - 4^2 = 25 - 16 = 9$$
$$6^2 - 5^2 = 36 - 25 = 11$$
$$12^2 - 11^2 = 144 - 121 = 23$$

You will notice that the final answer in each row is simply the sum of the two numbers we started with: for example, when we started with 5 and 4, we found that $5^2 - 4^2 = 9\ (= 5 + 4)$. Similarly $12^2 - 11^2 = 23\ (= 12 + 11)$.

Check for yourself that $10^2 - 9^2 = 10 + 9 = 19$. What are $30^2 - 29^2$ and $62^2 - 61^2$?

In the foregoing examples we chose *consecutive* numbers. Let us try again with numbers differing by *two*.

$$\begin{array}{ll} 3^2 - 1^2 = 9 - 1 = 8 & \text{but } 3 + 1 = 4 \\ 4^2 - 2^2 = 16 - 4 = 12 & \text{but } 4 + 2 = 6 \\ 5^2 - 3^2 = 25 - 9 = 16 & \text{but } 5 + 3 = 8 \\ 6^2 - 4^2 = 36 - 16 = 20 & \text{but } 6 + 4 = 10 \end{array}$$

We see that when the original numbers differ by 2, the answer is not just the sum of those numbers, but the sum of those numbers *multiplied by* 2.

Thus: $6^2 - 4^2 = (6 + 4) \times 2 = 20$

Check that $12^2 - 10^2 = (12 + 10) \times 2 = 44$. What is $100^2 - 98^2$?

If the original numbers differ by *three*, will we have to multiply their sum by 3? Let us try: $5^2 - 2^2 = 25 - 4 = 21$ and $5 + 2 = 7$

$6^2 - 3^2 = 36 - 9 = 27$ and $6 + 3 = 9$

It seems to work out: $5^2 - 2^2 = (5 + 2) \times 3$.

Check for yourself that $12^2 - 9^2 = (12 + 9) \times 3$.

The rule that emerges from all these experiments can be put like this: to find the difference between the *squares* of two numbers, multiply the *sum* of the two original numbers by their difference.

Try checking in some more cases:

$$10^2 - 5^2 = 100 - 25 = 75$$

by the rule, $10^2 - 5^2 = (10 + 5) \times (10 - 5)$

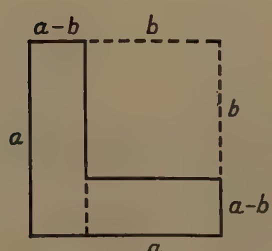
$$= 15 \times 5$$
$$= 75$$

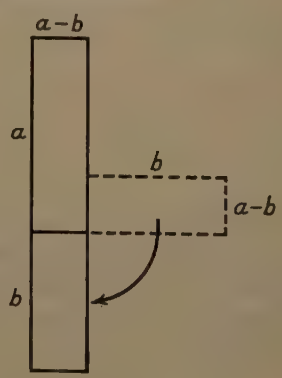
$$17^2 - 7^2 = 289 - 49 = 240$$

by the rule, $17^2 - 7^2 = (17 + 7) \times (17 - 7)$

$$= 24 \times 10$$
$$= 240$$

This is pretty convincing, but the rule is still not *proved*. To do that, let us look at a diagram, consisting of a square of side a with one corner cut away in the shape of a square of side b. The area left will then be $a^2 - b^2$.





The L-shaped area left is made into a rectangle by cutting away a portion and attaching it to the bottom, as shown. This new rectangle has length $(a + b)$ and breadth $(a - b)$, and its area is the same as that of the original L-shaped figure, in symbols:

$$a^2 - b^2 = (a + b) \times (a - b)$$

This proves our rule. If we multiply the sum of the numbers a and b by their difference, we get the difference between the *squares* of those numbers.

EXERCISE 11C

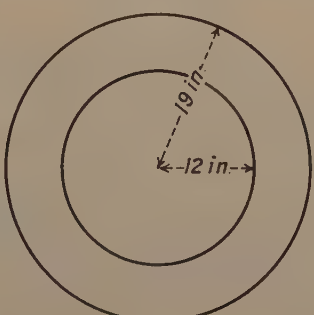
Use the rule given above to evaluate:

1. $11^2 - 9^2$
2. $68^2 - 58^2$
3. $26345^2 - 26344^2$
4. $6{\cdot}2^2 - 4{\cdot}2^2$
5. $17{\cdot}35^2 - 7{\cdot}35^2$

The Rule Applied to Other Problems

You will often find this rule very useful in arithmetic. For example, suppose it is required to find the area of the ring between two concentric circles, as in the diagram.

The area of the larger circle is $\pi \times 19^2$ (using the formula $A = \pi r^2$ for the area of a circle) and the area of the smaller one is $\pi \times 12^2$ in². Hence the area of the ring is



$$\begin{aligned} &\pi \times 19^2 - \pi \times 12^2 \text{ in}^2 \\ &= \pi(19^2 - 12^2) \text{ in}^2 \\ &= \pi(19 + 12)(19 - 12) \text{ in}^2 \\ &= \pi \times 31 \times 7 \text{ in}^2 \end{aligned}$$

If we take $\pi = \frac{22}{7}$, this is

$$\begin{aligned} &\frac{22}{7} \times 31 \times 7 \text{ in}^2 \\ &= 22 \times 31 \text{ in}^2 \\ &= 682 \text{ in}^2 \end{aligned}$$

As in the formulae for $(a + b)^2$ and $(a - b)^2$, the formula for the difference of two squares can be used for algebraic expressions as well as numbers. If we put $a = 3x$ and $b = 2y$ we have

$$(2x)^2 - (3y)^2 = (2x + 3y)(2x - 3y)$$
$$\text{or } 4x^2 - 9y^2 = (2x + 3y)(2x - 3y)$$

The latter form is most common. We have to be able to recognize that $4x^2 = (2x)^2$ and $9y^2 = (3y)^2$ in order to express $4x^2 - 9y^2$ in the two factors on the right. Similarly,

$$x^2 - 16y^2 = (x + 4y)(x - 4y)$$
$$p^2 - 9 = (p + 3)(p - 3)$$
$$25h^2 - 1 = (5h + 1)(5h - 1)$$

EXERCISE 11D

Use the formula for the difference of two squares to express the following in an alternative form:

1. $(a + 5)(a - 5)$
2. $(3x + 2)(3x - 2)$
3. $(2p + 9q)(2p - 9q)$
4. $y^2 - 36$
5. $z^2 - 1$
6. $4k^2 - 9$
7. $16x^2 - 121y^2$

Is the Problem Suited to the Formula?

The situations in which we need to apply the formula for the difference of two squares are usually easily recognized, but sometimes they are hidden by the presence of another factor. For example, we cannot at first sight use the rule for the expression $2p^2 - 8$
as neither term has an exact square root. We must spot the factor 2 and take it out, thus: $2p^2 - 8 = 2(p^2 - 4)$
and now the factor $(p^2 - 4)$ can be further factorized as $(p + 2)(p - 2)$, giving a final answer

$$2p^2 - 8 = 2(p + 2)(p - 2)$$

In the same way,

$$\begin{aligned} 27x^2 - 48y^2 &= 3(9x^2 - 16y^2) \\ &= 3(3x + 4y)(3x - 4y) \end{aligned}$$

EXERCISE 11E

Factorize:

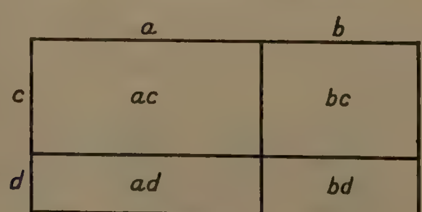
1. $3a^2 - 27$ 2. $5x^2 - 20y^2$ 3. $ab^2 - ac^2$ 4. $4px^2 - 9py^2$ 5. $hy^2 - h$

Multiplying Binomials Together

In the last three formulae we have seen how we can multiply certain brackets together, thus:

$$\begin{aligned} (a + b)(a + b) &= (a + b)^2 = a^2 + b^2 + 2ab \\ (a - b)(a - b) &= (a - b)^2 = a^2 + b^2 - 2ab \\ (a + b)(a - b) &= a^2 - b^2 \end{aligned}$$

Now we want a rule for multiplying *any* two brackets together. Again, the problem is soon solved with the aid of a diagram.



$(a + b)(c + d)$ can be represented as the area of a rectangle of length $(a + b)$ and breadth $(c + d)$, and the rectangle can then be sub-divided as shown.

The area can be seen to consist of four parts ac, ad, bc and bd: hence

$$(a + b)(c + d) = ac + ad + bc + bd$$

You should not try to memorize this formula: instead, analyze and remember the *process*. The second bracket, $(c + d)$, is multiplied, first by a

$$a(c + d) = ac + ad$$

and then by b

$$b(c + d) = bc + bd$$

and the two parts together produce the final result

$$ac + ad + bc + bd$$

In the same way,

$$\begin{aligned}(2a + 3b)(5x - 4y) &= 2a(5x - 4y) + 3b(5x - 4y)\\ &= 10ax - 8ay + 15bx - 12by\\ (3p - 2q)(y - 3z) &= 3p(y - 3z) - 2q(y - 3z)\\ &= 3py - 9pz - 2qy + 6qz\end{aligned}$$

EXERCISE 11F

Expand (this means multiply out):

1. $(h + 2k)(3x + y)$
2. $(2a + 4b)(3f - 6g)$
3. $(c - 7d)(2y + 3z)$
4. $(5a - 2b)(2s - 3t)$

Reversing the Process

The reverse process consists of taking an expression like

$$ax - 3ay + 2bx - 6by$$

and putting it back in its original brackets: in other words, factorizing it. To do this, we divide the four terms into two pairs, connected (here) by a + sign, thus: $(ax - 3ay) + (2bx - 6by)$

Next we spot the simple factors of each half separately:

$$a(x - 3y) + 2b(x - 3y)$$

and we recognize this as the first line in the multiplication of

$$(a + 2b)(x - 3y)$$

When the sign in the middle is minus, we must be careful to adjust the last sign at the second step. Consider:

	$2ap + 4aq - 3bp - 6bq$
Divide in pairs,	$(2ap + 4aq) - (3bp + 6bq)$ (Notice the sign change.)
Factorize the pairs,	$2a(p + 2q) - 3b(p + 2q)$
	$= (2a - 3b)(p + 2q)$

It is essential that the brackets which appear when we factorize the pairs should be the same in each half. If they are not identical, we cannot take the last step. For example, the line

$$a(x + 3y) + 2b(x - 3y)$$

is *not* the first line in the multiplication of

$$(a + 2b)(x + 3y)$$

and so these are *not* the required factors. Nor must we make wild guesses, such as

$$(a + 2b)(x + 3y)(x - 3y)$$

which is quite wrong. The simple answer is that there are *no* factors of this expression. If you reach such a line when answering a factorization question, then either someone is trying to catch you out, or you have made a mistake already, so check your work very carefully.

EXERCISE 11G

Factorize the following:

1. $ax + ay + 3x + 3y$
2. $cp - 2dp + 2cq - 4dq$
3. $x^2 + 4x - xy - 4y$
4. $hs - 5ht - 3ks + 15kt$

Factorization

Consider $\quad (a + 1)(x - y) = a(x - y) + 1(x - y)$
$$= ax - ay + x - y$$

If we took this answer out of the blue and tried to factorize it, we should work as follows:

Divide in pairs, $\quad (ax - ay) + (x - y)$
Factorize each pair, $a(x - y) + \quad ?$

We run into difficulty, because $x - y$ cannot be factorized. The clue, of course, comes from the multiplication above. $x - y$ can be written $1(x - y)$, although this can hardly be called factorization, and the work continues

$$a(x - y) + 1(x - y) = (a + 1)(x - y)$$

In the same way, we tackle

$$a^2 - ab - a + b$$

as follows: $\quad (a^2 - ab) - (a - b)$ (Note the signs.)
$$= a(a - b) - 1(a - b)$$
$$= (a - 1)(a - b)$$

There are other small difficulties which can arise. Take

$$px + py - 2y - 2x = p(x + y) - 2(y + x)$$

The two brackets are not exactly the same, but this does not really trouble us, as $x + y$ and $y + x$ come to the same thing, so there is no harm in rewriting the last line as

$$p(x + y) - 2(x + y) = (p - 2)(x + y)$$

However, the next one is not quite so straightforward:

$$px - py + qy - qx = p(x - y) + q(y - x)$$

Here $x - y$ and $y - x$ do *not* come to the same thing (for example, $8 - 3 = 5$, but $3 - 8 = -5$).

But notice that $+q(y - x) = qy - qx$

and $-q(x - y) = -qx + qy$

In other words, these two expressions give the same result (though the terms are in a different order). Thus we can alter $(y - x)$ into $(x - y)$ *provided that we also change the sign in front of the bracket.*

Continuing, we have

$$p(x - y) - q(x - y) = (p - q)(x - y).$$

Another example may help:

$$\begin{aligned} &ap - 2p - 2 + a \\ &= p(a - 2) - 1(2 - a) \\ &= p(a - 2) + 1(a - 2) && \text{(reversing the second bracket} \\ &= (p + 1)(a - 2) && \text{and changing the middle sign)} \end{aligned}$$

Finally, the formula for the difference of two squares may be involved, as here: $x^2 - y^2 + 2x - 2y = (x + y)(x - y) + 2(x - y)$

Treating $(x + y)$ as one quantity, this gives

$$\{(x + y) + 2\}(x - y)$$

or, more simply, $(x + y + 2)(x - y)$

Occasionally you will find that the terms have to be rearranged in order to find the factors. The expression

$$ac + bd + ad + bc$$

does not at first sight appear to be possible to factorize: but if we arrange it as

$$ac + ad + bc + bd$$

we find that it becomes a very easy example of the type we have been discussing. Try if for yourself.

EXERCISE 11H

Now try these. Factorize:

1. $ar - as + bs - br$

2. $x^2 - x + y - xy$

3. $ac - bc - b + a$

4. $p^2 - q^2 - px + qx$

5. $r^2 - 1 + rs - s$

Further Problems

These factor questions need some care and thought, but the multiplication process can be very easily extended.

Example:

$$\begin{aligned} &(x + 2)(2x^2 - x - 4) \\ &= x(2x^2 - x - 4) + 2(2x^2 - x - 4) \\ &= 2x^3 - x^2 - 4x + 4x^2 - 2x - 8 \end{aligned}$$

We now gather together like terms and give the final answer as

$$2x^3 + 3x^2 - 6x - 8$$

Similarly,
$$\begin{aligned}&(2x - y)(4x^2 - 4xy + y^2)\\ &= 2x(4x^2 - 4xy + y^2) - y(4x^2 - 4xy + y^2)\\ &= 8x^3 - 8x^2y + 2xy^2 - 4x^2y + 4xy^2 - y^3\\ &= 8x^3 - 12x^2y + 6xy^2 - y^3\end{aligned}$$

These are longer but no more difficult than the multiplications you have already done. They should not need much practice.

EXERCISE 11J

Expand:

1. $(2a - 3)(a^2 + 3a - 7)$
2. $(p + 3q)(3p^2 - 2pq - 2q^2)$
3. $(x + 1)(x - 2)(x - 3)$

(multiply two of the brackets together, then multiply the result by the third bracket)

Three-bracket Expressions

The last question in Exercise 11J shows how to multiply three brackets together. The reverse process, factorizing an expression which divides into three brackets, is generally too difficult for us to discuss here, but one or two simple cases are worth mentioning. Take the expression

$$a^4 - b^4$$

Using the difference of two squares, this can be factorized as

$$(a^2 + b^2)(a^2 - b^2)$$

Now the second bracket is itself the difference of two squares

$$a^2 - b^2 = (a + b)(a - b)$$

$$\text{Hence } a^4 - b^4 = (a^2 + b^2)(a + b)(a - b)$$

Here we have applied the formula for the difference of two squares *twice*.

Now consider
$$\begin{aligned}&x^3 + x^2 - x - 1\\ &= x^2(x + 1) - 1(x + 1)\\ &= (x^2 - 1)(x + 1)\end{aligned}$$

$$\text{Now } x^2 - 1 = (x + 1)(x - 1)$$

$$\begin{aligned}\text{So } x^3 + x^2 - x - 1 &= (x + 1)(x - 1)(x + 1)\\ &= (x + 1)^2(x - 1)\end{aligned}$$

Again we have applied a factorization process *twice*, though in this case the two processes have been different. It is important in *any* factorization problem to examine your answer to see if it can be factorized further.

EXERCISE 11K

Factorize completely:

1. $x^4 - 1$
2. $16p^4 - 81q^4$
3. $a^2b - b + 4a^2 - 4$

REVISION SUMMARY

ALGEBRA

LETTERS IN PLACE OF NUMBERS

Page

In Algebra, letters are used to stand for numbers.

Addition and Subtraction. **$5 + x$ means add to 5 the number that x stands for; $b - 3$ means subtract 3 from the number that b stands for.**

If there are C children in a cinema, and B of them are boys, how many of them are girls? 130

Multiplication and Division. **Multiplication signs are usually left out. $3c$ means 3 multiplied by the number c stands for.**

What is a shorter way of writing $x + x + x + x$? 131

$3a \times 4k = 12ak$

What is $2p \times 3q \times 4r$? 132

Division is usually expressed as a fraction. 6 divided by q is written $\frac{6}{q}$.

Like and Unlike Terms. **If two expressions contain the same combination of letters, they are called like terms: otherwise, they are unlike terms. Like terms can be added or subtracted, but not unlike terms.**

Simplify: $2c + 5c + 2d$. 135

Cancelling. **This is done exactly as in Arithmetic:**

$$\frac{6c}{2} = \frac{\overset{3}{\cancel{6}} \times c}{\cancel{2}} = 3c \qquad \frac{2a}{a^2} = \frac{2 \times \cancel{a}}{\cancel{a} \times a} = \frac{2}{a}$$

Simplify: (a) $\frac{5ab}{15ab}$ (b) $\frac{15p^2}{5p}$ 140

FORMULAE

Formulae **are rules or patterns, and to make a formula we have to find the pattern. If I travel 60 miles in 2 hours, my average speed is 30 mile/h, which is found by dividing 60 by 2. Whatever the figures, the average speed is always found by dividing the distance by the time. By working simple examples we find the general rule and express it in shortened form denoting the quantities involved by letters.**

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What is the formula for:

The average speed (v mile/h) at which a vehicle must travel in order to cover 240 miles in t hours? 145

The height (H ft) of a pile of 17 books, each A in thick? 145

***Brackets* save time. The contents of a bracket are to be treated as a single number. $2(a + 4)$ means: Add 4 to a and multiply the result by 2.**

When removing brackets, multiply each term in the bracket by the multiplier.

Using brackets, write the expression for:

The perimeter of a room 20 ft long $\times$ 12 ft wide. 147

Add 5 to y, and multiply the result by 2. 147

Remove the brackets in these examples:

$2(x + 3)$; $3(2a + 5c)$. 149

The time required to cook a joint of meat is $(15L + 20)$ minutes, where L represents the weight in pounds of the meat. If the joint has to be cooked for M minutes, what is the formula for M in terms of L? 150

Many useful formulae have already been worked out. To use them we replace the letters by the numbers in our particular problem.

If in the formula $R = \frac{1}{2}(24 - T)$, T is 10 inches, what is R? 154

EQUATIONS

***Equations* are sentences using mathematical shorthand which have two equal parts. $2n = 16$ is an equation.**

Make an equation from the following sentence. "A boy has a certain amount of pocket money. (Call this m pence.) He has twice this amount in his savings box and altogether he has 90p." 159

To solve an equation is to find the value of the unknown. (What is n if $2n = 16$?)

Collect the unknown on one side of the equation and the numbers on the other. Remember the rule, "Change the side, change the sign." Whatever you do to one side (add, subtract, multiply or divide) must be done to the other if the sentence is to remain true.

Solve these equations:

$$3b + 5 = 11. \qquad 162$$

$$2x + 4 = x + 7. \qquad 162$$

$$7\tfrac{1}{2} = \frac{x}{2} + 5\tfrac{1}{2}. \qquad 163$$

Page

Many problems can be solved with the aid of equations. Decide what you are asked to find, let a letter stand for this unknown, use the information in the question to construct a formula and then solve it.

Solve this problem with the aid of an equation:

A pet-shop owner bought some water snails at 4p each and found that 10 were dead. By selling the rest at 5p each he made a profit of 75p. How many did he buy? 165

The sign $\neq$ means "is not equal to", $>$ means "is greater than", $<$ means "is less than". Mathematical sentences can describe unequal quantities and still be true.

Are these sentences true or false?

$12 \neq 8 + 5 \qquad 8 > 7 \qquad 2 + 3 < 6$ 168

What values for x would make $x + 3 < 6$ a true sentence? 169

INDICES AND FRACTIONS

Index numbers are a shorthand method of writing numbers which are multiplied by themselves: $5^2 = 5 \times 5$,

$$7^3 = 7 \times 7 \times 7$$

$$\text{and } 11^5 = 11 \times 11 \times 11 \times 11 \times 11.$$

$3x^2$ means $3 \times x \times x$ and the number part (3) is called the coefficient of x^2. When different powers of the same number are multiplied together, the indices are added, for example,

$$3^4 \times 3^3 = 3^7,\ x^2 \times x = x^3,$$

and when the indices are divided the index numbers are subtracted,

$$\text{e.g., } a^4 \div a^3 = a.$$

Work out $3a^2 \times 5a^4$, 172

$\dfrac{6m^5}{3m^3}$ 173

When a number is squared its index number is doubled and the square root is found by halving the index number.

Work out $(2x)^2$, $\sqrt{49x^4}$ 173

The *factors* of a number are those numbers which divide into it exactly. (1, 2, 3, 6, 7, 14, 21 and 42 are the factors of 42.)

A *prime number* is divisible only by itself and 1, e.g. (2, 3, 5, 7, 11).

***Prime factors* are both prime numbers and factors.**

The *H.C.F.* of two or more numbers is the highest factor which they share.

Page

What is the H.C.F. of 40 and 60? Of $12x^2y$ and $20xy^2$? 175

A *multiple* of a number is any number into which it will divide exactly (12, 24, 36, 48, etc., are multiples of 12).
The *L.C.M.* of two or more numbers is the lowest multiple they share.

What is the L.C.M. of x^2y^3 and x^4y?
of $12a^2bc$ and $45a^3bc^3$? 176

Simplify: $\dfrac{2x^2y^2}{8xy^4}$ 177

Algebraic fractions obey the same rules as numerical ones. To multiply, the numerators are multiplied and then the denominators, cancelling first, if possible. To divide, invert the divisor and multiply.

Work out $\dfrac{2x^2}{y} \times \dfrac{5x^2}{3y^2}$ 179

$\dfrac{2a^2x}{3b^2x} \div \dfrac{4ax}{5bx}$ 180

Adding and subtracting fractions: Find an L.C.M. for the denominators, change each fraction to an equivalent fraction with the L.C.M. as denominator, and add (or subtract) these fractions.

Work out $\dfrac{x}{2} + \dfrac{x}{4y}$ 182

$x - \dfrac{y}{z}$ 183

A boy buys n ounces of sweets of which x ounces are chocolates and the rest is toffee. What fraction of the total is toffee? 184

FORMULAE AND PROBLEMS

In the formula $R = \frac{1}{2}(24 - T)$, R is the *subject* of the formula. *Changing the subject* means rearranging the formula so that T is on the left-hand side:

Remove the brackets, $2R = 24 - T$,

$$T = 24 - 2R$$

Rearrange $\dfrac{1}{7} + \dfrac{1}{9} = \dfrac{1}{x}$ so that x is the subject. 187

Rearrange $\dfrac{1}{u} + \dfrac{1}{v} = \dfrac{1}{f}$ so that f is the subject. 188

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Using equations to solve problems is of great importance:

First choose a letter to represent the unknown number you are asked to find, but remember that the letter stands for a number, not a quantity.

Next assemble the facts in an equation—don't mix units (e.g. feet and inches) but set out your equation in one sort of unit only.

Then solve the equation, but remember that this is not necessarily the answer to the problem.

Finally, check your answer in the original question, not in your equation.

Solve these problems, using equations:
Bob and John collect conkers. Bob has three times as many as John, but works out that if he gives John 6 of his he will then still have twice as many. How many did each have at first? 190

Mr. Bacon drove from Cambridge to York. He left at 8 a.m. and drove at an average speed of 30 mile/h which did not include the half-hour he stopped for coffee. His nephew, Augustus Egg, who drove a sports car, left at 9 a.m. and took a route which was 10 miles longer. He averaged 40 mile/h, did not stop for coffee and arrived $\frac{1}{2}$ hour before his uncle. How far was it from Cambridge to York by the direct route? 192

DIRECTED NUMBERS

Plus and minus signs can be instructions (add or subtract), or adjectives, representing movements, or debts, or quantities less than zero.
Adding a minus or negative number is the same as subtracting a positive number.

The Rule of Signs is:

$+(+n) = +n; \quad +(-n) = -n;$
$-(+n) = -n;$ **and** $-(-n) = +n.$

Work out $(+4) + (-2) \quad (+2) - (-3)$ 200
$(+2) - (+3) \quad (-2) - (-3)$ 201

When directed numbers are *multiplied* and *divided*, like signs give a plus answer, unlike signs give a minus answer.

Work out $(+5) \times (-4) \quad (-5) \times (+4)$
$(-5) \times (-4) \quad (+5) \times (+4)$ 204
$\dfrac{-8}{-4} \quad \dfrac{-8}{+4}$ 205

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When removing a bracket the signs remain unchanged if it has a plus sign in front of it, but with a minus sign in front, *all* the signs inside are changed.

Remove the brackets in these expressions:

$-(b^2 + 3b - 4)$

$-2(a^2 + 5a - 6)$

$-x + 3 - 5(x - 2)$ 207

Complete the following $a - 2b - 3c = a - (\ldots\ldots)$ 208

Equations may involve the use of directed numbers.

Solve the equations:

$-2(x + 3) = 12$

$3(x + 4) - 2(2x - 3) = -2(2x - 18)$ 209

GRAPHS

Graphs are used to present numbers in diagrammatic form for purposes of comparison. There are many different types.

Picture graphs. **The numbers being presented are represented by pictures resembling the objects concerned: the height of each picture is proportional to the number it represents, and a vertical scale enables the reader to estimate this number.**

Show in a picture graph the populations of the cities given below. (The figures represent millions of inhabitants.)

City A	3·48	City E	8·35	
City B	8·53	City F	7·80	
City C	2·85	City G	4·80	211
City D	1·97			

Pie charts. **Here a circle is divided into sectors whose areas are proportional to the percentages into which some given quantity is divided. Details are given in each sector or around the circumference.**

Show in a pie-chart the division of a certain country's building investment on various items as follows:

Factories	33%	Farms	6%	
Fuel installations	15%	Transport	18%	
Schools, hospitals, etc.	7%	Others	5%	
Housing	16%			213

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Ideographs. **A certain number or value of an item is shown by a symbol called an isotype.**

By drawing several such symbols the total number or value of that item can be shown.

Show the distribution of exports below by means of an ideograph. Choose symbols to represent 1 million pounds' worth of each item. The figures represent millions of pounds' worth of goods exported.

Vehicles 23	Textiles 36	Machinery 25
Chemicals 9	Electrical goods 8	

214

Column graphs. **These are just like picture graphs, except that instead of pictures the numbers are represented by rectangular columns of various heights.**

Illustrate the temperature measured at different times of day, as under, by means of a column graph.

2 a.m.	39° F	8 a.m.	40° F	2 p.m.	51° F	8 p.m.	46° F
4 a.m.	35° F	10 a.m.	42° F	4 p.m.	52° F	10 p.m.	43° F
6 a.m.	38° F	12 noon	46° F	6 p.m.	50° F	12 mdt.	40° F

215

Curve graphs. **These are usually plotted on squared paper. A horizontal axis is marked with a scale to represent the quantities selected, and a vertical axis is marked with a scale to represent the quantities measured or calculated.**

The graph is constructed as a column graph, but only the top of each column is marked with a point. If both quantities are continuously changing, these points are joined up by a curve (or straight line) to assist estimation of intermediate values. Axes and scales should be chosen to make full use of the paper available and it is not necessary to start either scale at zero.

Draw a curve graph to show how £100 invested at 4% gradually accumulates in value as time passes, using these figures:

Number of years	0	5	10	15	20	25
Amounts in pounds	100	122	148	180	219	267

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Histograms. **If the same quantity is measured in a large number of different cases (such as the heights of all the pupils in a school), it will be found that, for example, 18 pupils have a height between 5 ft 2 in and 5 ft 4 in.**

We say then that the ***frequency*** **of the class-interval 5 ft 2 in-5 ft 4 in is 18.**

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A histogram shows the class-intervals along a horizontal axis and represents the frequencies by rectangles whose bases are the class-intervals and whose heights are the frequencies.

Draw a histogram to show the distribution of marks in an examination from these figures:

Class-interval	*Frequency*	*Class-interval*	*Frequency*
0 – 10	1	51 – 60	22
11 – 20	3	61 – 70	17
21 – 30	5	71 – 80	15
31 – 40	3	81 – 90	1
41 – 50	14		

221

FRACTIONS AND COMPOUND BRACKETS

A binomial is a number with two parts,

e.g. $(a + 3)$, $(x^2 + 5)$.

Some calculations can be made easier by using the form

$$nx + ny = n(x + y).$$

and $$nx - ny = n(x - y).$$

Work out by this method:

$8^2 + 8 \times 4$, $\quad 18 \times 11 - 18$, $\quad 70 \times 69$, $\quad 26 \times 1\frac{3}{4}$. 223

$nx + ny = n(x + y)$ is an identity where each side is an expanded or abbreviated form of the other.

Square roots. **Since both $(-n) \times (-n)$ and $(+n) \times (+n) = +n^2$, the square root of n^2 is $\pm n$, but the square root sign $\sqrt{}$ is used only for the positive square root.**

What are the values for x in the equation

$$(x + 3)^2 = 36?$$ 226

To add or subtract fractions with binomial numerators, find the L.C.M. of the denominators and change each fraction to an equivalent fraction with that as denominator. They can then be added in the normal way.

Simplify: $\dfrac{x + 3}{4} \times \dfrac{x + 2}{5}$ 226

$\dfrac{a + 3b}{2b} - \dfrac{a - 4b}{5a}$ 227

To solve equations involving fractions **it is necessary first to clear the fractions by multiplying each term by the L.C.M. of the denominators. The equation can then be solved and checked in the normal way.**

Page

Solve these equations:

$$\frac{3a}{8} - \frac{1}{2} = \frac{a}{6}$$ 229

$$\frac{1}{a} - \frac{1}{2a} = \frac{1}{5}$$ 230

$$\frac{x+2}{10} + \frac{x+3}{5} = \frac{7}{8}$$ 230

If we want to multiply an expression like $x(a-1)$ by $2b$ we could use *compound brackets* thus:

$$2b[x(a-1)]$$

or the vinculum, $2b(x\overline{a-1})$.

If it is necessary to simplify such an expression, remove the inner brackets first.

Simplify: $\frac{1}{2}[2(x-y) - 5(x+2y)]$ 232

Solve the equation: $2a - (3a+4) + [(2a-1) - (3-6a)] = 13$ 233

MORE ABOUT GRAPHS

Graphs are often used to illustrate formulae. When calculating the required values the work should be tabulated.

Copy and complete the following table of values for the formula $s = 80t - 16t^2$ for values of t from 0 to 6.

t	0	1	2	3	4	5	6
$80t$	0	80	160				
$-16t^2$	0	-16	-64				
s	0	64	96				

234

Draw the graph of the formula 242

***Travel graphs.* When dealing with constant speeds, travel graphs represent journeys by straight lines: the time axis is horizontal and the vertical axis represents distance from a given place, NOT necessarily distance travelled.**

Draw travel graphs for the following journeys on the same diagram:

(*a*) A man sets out walking along a road at 10 a.m. walking at 4 mile/h.

(*b*) A boy leaves the same spot at 10.30 a.m., cycling at 10 mile/h.

(*c*) A car leaves a town 20 miles away at 11 a.m. and travels to meet them at 20 mile/h. 238

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Conversion graphs **are straight lines used to convert measurements from one unit to another. If a set of marks ranges from 11 to 92, and we wish them to range from 20 to 80, we draw a graph with the point 11 on one scale corresponding to 20 on the other, and similarly with 92 and 80, then join these two points with a straight line.**

Draw a graph to scale marks from a range 0 to 130 to a range 0 to 100. 239

Linear Functions. **When the graph illustrating the connection between two quantities is a straight line, we say that each is a linear function of the other. The formula for any linear function can be put in the form $y = mx + c$.**
The number m in any particular formula is the *gradient*, and the number c tells us the point where the line *cuts the y axis.*

The gradient is the ratio of the rates at which y and x increase.

SIMULTANEOUS EQUATIONS

Solving simultaneous equations means finding values of the unknown letters which satisfy two different equations at once.

Graphical method: **Draw on the same diagram the graphs of each equation. The values of the unknowns at the point of intersection give the solution.**

Solve the equations: $x + y = 8$
$y - x = 4$ 250

Substitution method: **If in one of the equations it is easy to express one letter in terms of the other, we may replace this letter in the other equation by its equivalent expression.**

To solve

$$y + 7 = 4x \qquad (1)$$
$$2x = 5y - 8 \qquad (2)$$

write equation (1) as

$$y = 4x - 7$$

and by substitution in equation (2) we get

$$2x = 5(4x - 7) - 8,$$

from which we may find x.

Solve the equations $x = 3 - 2y$
$4x - 2y = -28$ 256

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Elimination method: **By suitable multiplication of each equation we can ensure that one letter has the same value in each equation but with opposite signs. Adding the two equations now eliminates that letter.**

To solve $3x - 5y = 2 \quad (1)$

$2x - 7y = -6 \quad (2)$

multiply (1) by 2 and (2) by -3, to get

$$6x - 10y = 4$$
$$-6x + 21y = 18$$

addition now gives $11y = 22$

Solve the equations $3x - y = 1$
$5x + 2y = 20$ 259

Problems: **Simultaneous equations usually arise from problems which give the data in two sentences. Each sentence must be translated into a different equation.**

A handful of coins, five in number, consists of 5p and 10p pieces. The coins are worth 35p altogether. How many are there of each? 263

SIMPLE FACTORS. SQUARES AND PRODUCTS

Factorization is the process of finding, in as simple a form as possible, a number of expressions which when ***multiplied together*** **make a given expression. The complete factorization of $6x^2 - 3xy$ is $3x(2x - y)$. The answer $3(2x^2 - xy)$ would** ***not*** **be correct because the factor x has not been isolated.**

Factorize $4ab - 6b^2$, $c^2 + c$ 267

Squares: **these formulae must be known:**

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$

Expand $(4x + 3y)^2$, $(3a - 5)^2$ 269

Difference of two squares: $a^2 - b^2 = (a + b)(a - b)$

Factorize $4x^2 - 9y^2$, $25h^2 - 1$ 271

It may be necessary to isolate an obvious factor before using the formula: thus

$$3a^2 - 12 = 3(a^2 - 4) = 3(a + 2)(a - 2)$$

Factorize $27x^2 - 48y^2$ 272

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Multiplication: When two expressions in brackets are multiplied together, the second bracket must be multiplied by each term in the first bracket. Thus:

$$(2a + b)(3p - 4q)$$
$$2a(3p - 4q) + b(3p - 4q)$$
$$6ap - 8aq + 3bp - 4bq$$

Expand $(2a + 3b)(5x - 4y)$
$(3p - 2q)(y - 3z)$ 273

Factors of four-term expressions. To do this we reverse the process of multiplication. There are many small points to watch for, illustrated in the following examples:

$$2ax + 3bx + 4a + 6b = x(2a + 3b) + 2(2a + 3b)$$
$$= (x + 2)(2a + 3b)$$
$$2a + 2ay - 3bx - 3by = 2a(x + y) - 3b(x + y) \text{ (note the sign change)}$$
$$= (2a - 3b)(x + y)$$
$$5px - 10qy - 4py + 2qx = 5px + 2qx - 10qy - 4py \text{ (changing the order)}$$
$$= x(5p + 2q) - 2y(5q + 2p)$$
$$= (x - 2y)(5q + 2p)$$
$$ax - ay + x = a(x - y) - 1(y - x) \text{ (note the unusual factor 1)}$$
$$= a(x - y)\,1(x - y) \text{ (double sign}$$
$$= (a + 1)(x - y) \text{ change)}$$
$$p^2 - q^2 + p - q = (p + q)(p - q) + 1(p - q)$$
$$= (p + q + 1)(p - q)$$

Factorize $ax - 3ay + 2bx - 6by$ 273
$a^2 - ab - a + b$ 274
$px - py + qy - qx$ 275
$x^2 - y^2 + 2x - 2y$ 275

Double factorization: The result of one of the factorization processes already discussed may itself be factorized. For example:

$$x^4 - 1 = (x^2 + 1)(x^2 - 1)$$
$$= (x^2 + 1)(x + 1)(x - 1)$$

Factorize $x^3 + x^2 - x - 1$ 276

GEOMETRY

CHAPTER 1

POINTS AND THEIR LOCATION

GEOMETRY is concerned with points. It is not easy to say exactly what is meant by "a point." Some think of it only as a mark on a piece of paper; but any object—a railway train, a ship at sea, London, even the Earth itself —may be regarded as a point. The important thing is to be able to recognize a point when we read about one. When we think of the distance from London to Birmingham we are thinking of these two cities as points. When we think of the distance from Buckingham Palace to the Marble Arch, these two things count as points, but London does not then count as a point; it is instead, a large collection of points.

In the same way the Earth may be thought of as a point, or as a large collection of points. To count as a point, the thing of which we are thinking must be so small that its size can be neglected in comparison with its distances from the other points of which we are thinking. The paper on which you are going to write and draw may be regarded as either a point or as a large collection of points.

Wherever we are, we are surrounded by other objects (points). As geometers our only interest in these objects is *where they are*, i.e. their *positions.* An artist might be interested in the colour of an object, a chemist in its material, a salesman in its value. A geometer is interested only in its position. We shall represent ourselves and other objects by points on paper. We are not concerned with what these other objects are; they may be cows in a field, ships at sea, towns in a county, fielders in a game of cricket. We think of all these as points and can represent them on our paper. We represent ourselves by the point O and another object by the point A. The question we have to answer is "Where is A?" To answer this question we need to understand what is meant by distance and direction. We shall then be able to answer the question with the aid of numbers.

The distance OA is measured by counting the number of steps to take us from O to A. The step may be of any convenient size, but all the steps we take must be the same size. A convenient size for measurement on our paper is an inch (or a centimetre).

In the garden a stride will usually do for rough work. The distance between two points will not usually be an exact number of steps and there will generally be a bit left over which is not a full step. We get a more accurate measurement by counting the number of tenths of a step in this bit.

The ruler which we use is marked (and numbered) in inch-steps and each inch is divided into tenths. With a ruler we should obtain a correct count of the whole inches and tenths in a distance, and we can guess the number of hundredths in the left-over bit to obtain a very accurate measurement of the distance OA. If OS is the length of a step, then the symbol $\frac{OA}{OS}$ is used for the number of steps (of size OS) contained in OA. In the example (Fig. 1),

$$\frac{OA}{OS} = 2{\cdot}67.$$

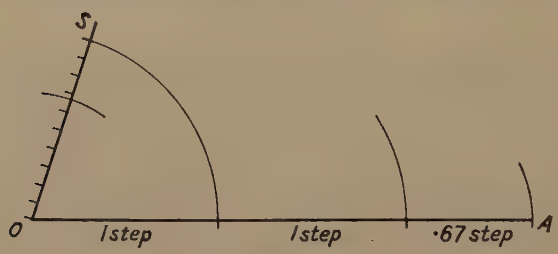


FIG. 1

Direction and Angle

To understand the idea of direction it is first necessary to learn something about angles. If there are two objects A and B and ourselves, as usual, at O, then the lines OA and OB form an angle. O is called the vertex of the angle and OA, OB its arms. The angle will be called $A\hat{O}B$. (Note that the vertex is the middle letter of the three.) An angle $A\hat{O}B$ is measured by the amount of turn we have to make when we turn from looking along OA and look along OB. If A, O and B lie (in that order) along a straight line, the angle AOB is said to be a straight angle. If we draw another line, OC, we make two more angles, $A\hat{O}C$ and $C\hat{O}B$. Such angles are called "adjacent." This word has nothing to do with their sizes. It is used for two angles which share an arm (OC) and whose other arms (OA and OB) are in a straight line.*

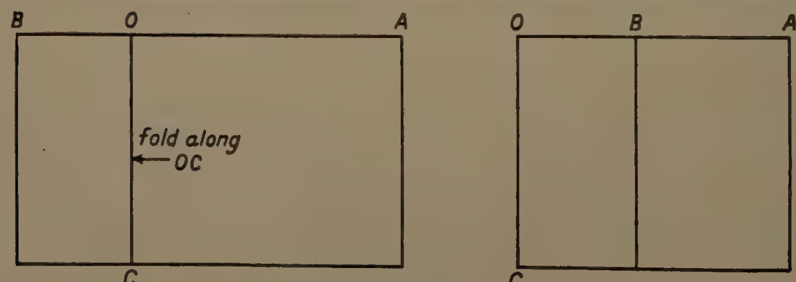


FIG. 2

* Angles which share an arm are frequently referred to as "adjacent," even when the other arms are *not* in a straight line.

If we now fold the paper over by making a crease along OC we shall usually find that one of these adjacent angles is bigger than the other. But it may happen that the two angles are the same size, and OB will lie exactly along OA when we fold the paper in this manner. (The diagram (Fig. 2) shows this happening and you should try folding a piece of paper in this way to see how easily it can be done.)

Kinds of Angles

When two adjacent angles are equal, then each is called a RIGHT-ANGLE. When they are not equal, the larger is said to be OBTUSE and the smaller is said to be ACUTE. Thus an obtuse angle is one which is larger than its adjacent angle; an acute angle is one which is less than its adjacent angle; and a right-angle is one which is equal to its adjacent angle.

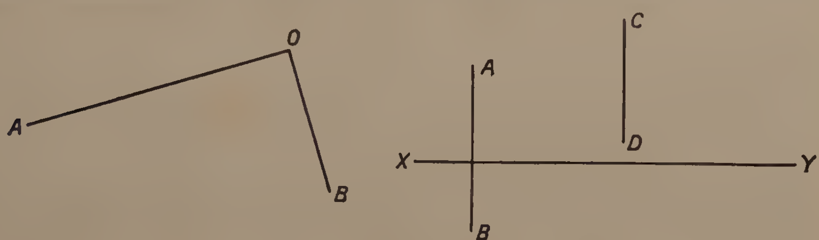


FIG. 3

Two lines which form a right-angle are said to be "perpendicular (to one another)." The words in brackets are usually omitted, but it should never be forgotten that they are there. The word "perpendicular" should be used very carefully. It is nonsense to talk of a "perpendicular" line or to say "a line is perpendicular"; but it makes sense to say "two perpendicular lines" or "a pair of perpendicular lines." It also makes sense to say "OA is perpendicular to OB" and "AB and CD are perpendicular to XY" (Fig. 3). Note that the last sentence would have an entirely different meaning if the words "to XY" were omitted.

The right-angle, then, is a ready-made standard for measuring angles, but as most of the angles we deal with will be less than two right-angles, it is too large for this purpose and we have to subdivide it. The Babylonians originally had the idea of dividing the right-angle into 90 (equal) parts, each called a degree (°) and this has worked well for thousands of years and it is still used. (The present move into decimalization fortunately makes no suggestion that we should divide the right angle into 100 parts instead of 90; it is difficult to imagine a less suitable unit of angle than a hundredth part of a right angle!)

The degree enables us to measure our angles in whole numbers. Navigators of aircraft and ships can set their courses accurately enough without fractions of a degree, but surveyors and astronomers need finer divisions and for their purposes the degree is sub-divided into 60 equal parts, called minutes (′) and it is sometimes necessary to subdivide the minute into 60 equal parts called seconds (″). We, however, will make do with degrees, and if we need greater accuracy than this, we shall try to guess the number of tenths of a degree just as we did with hundredths of an inch when measuring lengths.

Measurement of Angles

The protractor is an instrument for measuring angles on paper, to the nearest degree, but it would not be of much use in measuring the angle between the lines joining my eye to the two chimney-pots which I can see from my window as I write. There are other instruments for this purpose, but as we shall be measuring angles only on paper, the protractor will do for us.

The protractor with straight edges has three of its edges marked in degrees. The fourth side is placed along one of the arms of the angle we wish to measure with the centre of this side on the vertex of the angle. The size of the angle can then be read off at the point where the other arm of the angle crosses the scale. The protractor with one straight side and one curved is used in a similar manner, but the point to be placed at the vertex of the angle is not usually at the centre of the straight side. It is at the centre of the line which joins the scale points marked 0° and 180°. We can thus measure any angle to the nearest degree, or, if we need to be more accurate, to the nearest tenth of a degree.

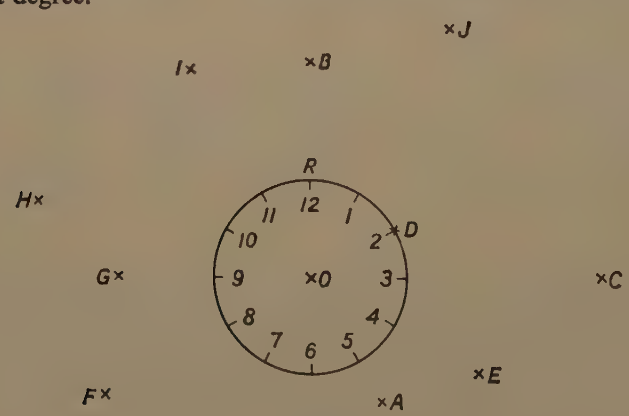


FIG. 4

Fig. 4 represents our position, O as usual, and a number of other objects $A, B, C, D, E, F, G, H, I, J$. We can now answer the question "Where is A?"

We choose a point, R, as our "reference point" and we surround ourselves with an imaginary clock-face, marked in hours with 12 o'clock on OR. If we now take OR as the length of a step, the position of a point can now be given by the hour-line on which it lies and the number of steps required to reach it.

Example: A is 1·5 steps at 5 o'clock; G is 2 steps at 9 o'clock, etc. All the points except H have been cunningly chosen so that they lie on hour-lines, but H is "between 9 and 10 o'clock."

This method of giving position is good enough for rough purposes, but for more accurate purposes we require to subdivide the "hours." If we divide the space between each hour into 30 (equal) parts, each part will be 1° and so we can give directions to the nearest degree.

The position of the point H can now be given more accurately. Note that 5 o'clock is also called 150° and we shall now dispense with hours and use degrees instead. (You may imagine the diagram to represent any set of objects you wish; it was drawn imagining O as the batsman and B as the bowler in a game of cricket; the other points represent fielders. The wicket-keeper is omitted; he is near enough to O to be regarded as the same point.)

Question: Where is F? *Answer:* $\dfrac{OF}{OR} = 2{\cdot}5$, $R\hat{O}F = 240°$.

(Note that the angle, or amount of turn, is always given clockwise.)

EXERCISE 1A

Answer in a similar manner the questions: Where are B, C, D, E, H, I, J?

The answer to the question: "Where is A?" is called the *location* of A and the verb "to locate" is used in connexion with the process of finding an answer to this question. The words are derived from the Latin word *locus* meaning "a place." It is very important to know the exact meanings of these words when they are used. Note that we cannot begin to locate a point until we have fixed O (Ourselves) and R (the Reference point). If there are no instructions how these are to be taken, we can take them where we please. If R is north of O, $R\hat{O}A$ is called the *bearing of OA*.

When we are given the location of a point we are able to mark its position on paper. This process is called "plotting" the point.

EXERCISE 1B

Taking O in the centre of your paper and R 1 in. above it, plot the other points mentioned in the questions and label each point with its letter:

1. $\dfrac{OA}{OR} = 2$; $R\hat{O}A = 30°$ **2.** $\dfrac{OB}{OR} = 2$; $R\hat{O}B = 90°$

3. $\dfrac{OC}{OR} = 3$; $R\hat{O}C = 150°$

4. $\dfrac{OD}{OR} = 2{\cdot}6$; $R\hat{O}D = 125°$

5. $\dfrac{OE}{OR} = 1$; $R\hat{O}E = 300°$

6. $\dfrac{OF}{OR} = 1{\cdot}6$; $R\hat{O}F = 215°$

7. Locate X, the point where DF meets OC. (Your answer should be given in the form $\dfrac{OX}{OR} = \ldots$; $R\hat{O}X = \ldots$)

8. Locate the point Y, where AC meets OB.

9. Locate Z, where BF meets OD.

(Start with O and R as before, but on a fresh sheet of paper.)

10. Plot the point A, given $\dfrac{OA}{OR} = 2$; $R\hat{O}A = 60°$. Plot the point B, given $\dfrac{OB}{OA} = 1$; $A\hat{O}B = 60°$. Plot C, given $\dfrac{OC}{OR} = 1$; $R\hat{O}C = 120°$. Locate D, the point where BR and AC meet.

(The answer to this should be in the form

$$\frac{OD}{OR} = \ldots, \; R\hat{O}D = \ldots)$$

In No. 10, observe that we started with O and R; we then plotted A with reference to O and R; we then plotted B with reference to O and A; we then plotted C with reference to O and R; and finally we plotted D from its location with reference to A, B, C and R. We thus see that there are many ways of locating a point.

When A has been plotted it may be used to locate a later point—in the example given it was used to locate B and D. When B has been plotted it may also be used to locate a later point. There is no limit to the number of points which may appear in a figure and any point already in the figure may be used to locate a new point. We also notice that the location of a point is complete when we have two facts relating it to other known points. (The *two* facts are clear in the cases of A, B and C; the *two* facts which locate D are: (a) D lies on BR, and (b) D lies on AC.)

(Use a fresh O and a fresh R for each of the questions 11-15.)

Given $OR = 1$ in, plot A and B from the locations:

11. $\dfrac{OA}{OR} = 2$, $R\hat{O}A = 60°$; B lies on RA and $R\hat{O}B = 30°$.

12. $R\hat{O}A = 110°$, $O\hat{R}A = 20°$; $\dfrac{OB}{OA} = 2$, $A\hat{O}B = 40°$.

13. $\dfrac{OA}{OR} = 2$, $\dfrac{RA}{RO} = 2$; B lies on the circle centre A passing through O, and also on the line RA.

14. $R\hat{O}A = 70°, \dfrac{OA}{OR} = 2; O\hat{R}B = 110°, O\hat{A}B = 110°.$

15. $\dfrac{OA}{OR} = 1, R\hat{O}A = 90°; RB = RO, AB = AO.$

Partially Located Points

We have seen that each new point requires two facts to locate it. If we are given only one fact about the new point it will not be possible to plot the point; but the *possible positions* of the point can be marked.

Suppose, for example, that we have started with O and R as usual, and the only fact we know about the new point, A, is that $R\hat{O}A = 90°$. If we now draw the line through O making 90° with RO, this line will contain all the *possible positions of A*.

Or, again, if we know only that $\dfrac{OB}{OR} = 1$, then the circle with centre O which passes through R will contain all the *possible positions of B*.

When a point is partially located, its possible positions will form a pattern, and this pattern is called the *locus of the point*. In the two examples given above, the *locus of A is the straight line through O perpendicular to OR* and the *locus of B is the circle with centre O passing through R*. We shall use the word *fixed* to denote a completely located point, and the word *variable* to denote a partially located point. In the examples given above, O and R are fixed points; A and B are variable.

Locus, then, is something which belongs to a variable point, i.e. a point with insufficient information to locate it. When we talk about the locus of A we mean the pattern of possible positions of A, and we cannot begin to talk about this pattern (locus) until we have some information about A. In the two examples above we were given some information about A (and B) and we were then able to describe, in words, the locus of A (and B). We could also draw a diagram containing the two fixed points, O and R, and the locus of A and the locus of B. The diagram would not contain either A or B since these are not completely located; but it would contain the locus of A and the locus of B, and it could contain the locus of any other partially located point.

The plural of the word "locus" is "loci," and this word is used when we are talking of more than one locus, e.g. the diagram of the last sentence would contain the loci of A and B. Even when we are given a completely located point to plot we can use the idea of locus, for each piece of information about the point leads to a locus, so that the point has two loci (one for each piece of information) and the point can then be plotted as the intersection of these two loci.

Example. The point A is located by the two facts: (1) $OA = 1$ in; (2) $RA = 1{\cdot}8$ in. O and R are two fixed points 1 in apart. The locus of A from fact (1) is the circle centre O, radius 1 in; the locus of A from fact (2) is the circle centre R, radius $1{\cdot}8$ in; and the position of A is the point (in this case, two points) belonging to both loci (Fig. 5).

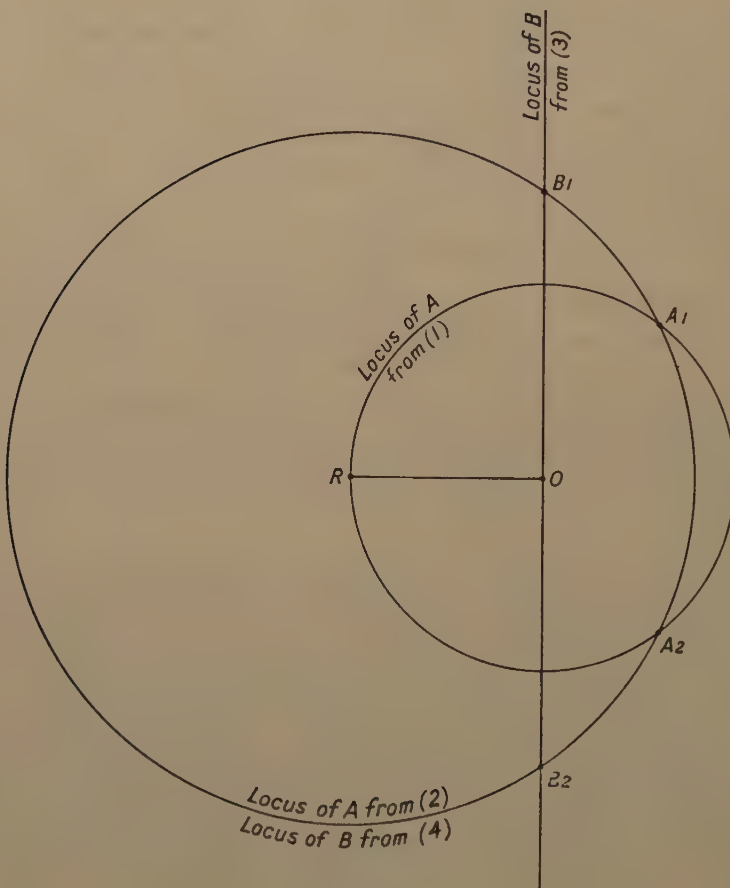


FIG. 5

Example: The point B is located from the two facts: (3) $ROB = 90°$; (4) $RB = 1{\cdot}8$ in. O and R are two fixed points 1 in apart as before. The locus of B from fact (3) is the line through O perpendicular to RO; the locus of B from fact (4) is the circle centre R, radius $1{\cdot}8$ in; and the position of B is the point (in this case also, two points) belonging to both loci.

These two examples are illustrated in the same diagram, which contains the fixed points O and R, and the loci of:

A from fact (1), A from fact (2), B from fact (3) and B from fact (4). Note that two of these four loci are the same curve, viz. the circle centre R, radius 1·8 in. A has two positions, marked in the diagram as A_1 and A_2. B also has two positions, marked B_1 and B_2. This method of marking a point is useful when the point has more than one position.

Example. Given O and R are 4 cm apart, $OA = RA$, what is the locus of A?

In this case we can plot as many possible positions of A as we wish, but we could never plot all of them. One possible position of A is an intersection of the circles centres O and R with radius 4·8 cm and by choosing other radii (plural of radius) we could plot as many as we wished. By plotting a number of such possible positions we can see that the locus of A is the line passing through the mid-point of OR and perpendicular to OR. This line is called the perpendicular bisector of OR. Our answer to the question, then, is:

When $OA = RA$, the locus of A is the perpendicular bisector of OR.

In this example we have been given that $OA = RA$. Another way of giving this fact is to say that A is equidistant from O and R. Our answer to the question might have been given differently:

When A is equidistant from O and R, the locus of A is the perpendicular bisector of OR. We could answer the question without even mentioning A:

The locus of a point equidistant from O and R is the perpendicular bisector of OR. In this form we have a result of considerable importance, for any two given points could take the place of O and R in this sentence and we shall often use the fact that a point is equidistant from two others.

In what follows, A, B and C may be plotted anywhere on the paper.

Plot a point equidistant from A and B.

To follow this instruction we draw a circle with centre A; we then draw a circle of the same radius with centre B, and mark an intersection with the other circle. In practice, we should not draw the complete circles, but enough of each to give the intersection. Such a part of a circle is called an arc. (If the circles do not intersect we have to try again with a larger radius.)

Draw the perpendicular bisector of AB.

To follow this instruction we plot a point, P, equidistant from A and B; we then plot a second point, Q, also equidistant from A and B; we then draw the line through P and Q, and this line will be the perpendicular bisector of AB.

(In following these instructions P and Q should not be taken too close together; the farther apart they are the more accurate the drawing.)

Note that if PQ meets AB at M, then M will be the *mid-point* of AB.

Note also that in following these instructions we have used only ruler and compasses, and the ruler has not been used for measurement, but only for making a straight line. Such instructions are called ruler-and-compass constructions.

Plot the point equidistant from A, B and C.

If we name the required point X, then X is located from the two facts:

(1) X is equidistant from A and B;

(2) X is equidistant from B and C.

(X is also equidistant from A and C but this third fact is not necessary.) The locus of X from fact (1) is the perpendicular bisector of AB; the locus of X from fact (2) is the perpendicular bisector of BC. If we draw these two loci, X can be plotted at their point of intersection. (If the perpendicular bisector of AC is now drawn it should also pass through X and by drawing it we can see how accurate our drawing is.)

Draw the circle through A, B and C.

We construct the point, X, equidistant from A, B and C. The circle with centre X passing through A will also pass through B and C and is thus the circle required. This circle is called *the circum-circle of ABC.*

Draw the perpendicular from A to BC (i.e. the line through A which makes right-angles with BC).

Method 1: With centre A and any radius draw a circle to cut BC at G and H. Mark a point, K, equidistant from G and H. Then AK is the required perpendicular. Note (1) the complete circle need not be drawn; an arc will be sufficient: (2) if the circle does not cut BC take a larger radius.

Method 2: With centre B draw the circle through A. With centre C draw the circle through A. Mark P, the other intersection of these circles. Then AP is the required perpendicular.

Of these two methods, the first will work wherever A, B and C may be. The second method uses fewer points, but fails when A is in the line BC or when the line BC is too near to the edge of the paper.

EXERCISE 1C

1. Draw a line PQ of length 3 in.
Plot a point R such that $PR = QR = 2\frac{1}{2}$ in.
Draw the perpendicular bisector of PR.
Draw the circle passing through P, Q and R.

(To check your drawing, measure the radius of this circle, which should be 1·56 in.)

2. Taking O and R in any convenient positions, plot the point A such that $\frac{OA}{OR} = 1$ and $R\hat{O}A = 120°$. Draw the circum-circle of OAR.

Is the radius of this circle greater or less than OR?

3. Plot B such that $\frac{OB}{OR} = 2$ and $R\hat{O}B = 90°$. Draw the perpendicular from O to RB.

Plot a point C whose distances from O and R are equal to the length of this perpendicular.

4. Use Method 2 to draw the perpendiculars from (a) S to TU; (b) T to US; and (c) U to ST (S, T and U may be taken anywhere.).

5. Draw any triangle ABC and mark a point P inside the triangle. Use Method 2 to draw the perpendiculars from P to the sides of this triangle.

6. Plot O and R 10 cm apart. Plot a point D such that $R\hat{O}D = 40°$. Draw the perpendicular from R to OD and measure the length of this perpendicular (6·4 cm).

Parallel Lines

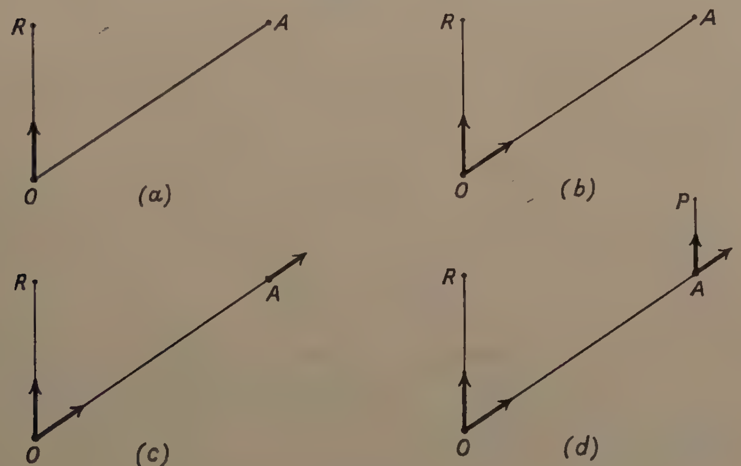


FIG. 6

Fig. 6*a* shows an observer O looking along the arrow towards a reference point, R.

Fig. 6*b* shows him looking along a new arrow towards an object, A.

Fig. 6*c* shows him after he has walked to A, still looking along the same line.

Fig. 6*d* shows him after making a left turn of the same size as his original right turn ($R\hat{O}A$).

Now if, in Fig. 6*d*, he makes too large a turn, he will be looking *towards* the line OR. If he makes too small a turn, he will be looking *away from* the line OR. But if he makes exactly the same turn, he will be looking neither towards nor away from OR; the line along which he is looking is said to be *parallel to OR* and the lines OR and AP are said to be *parallel*.

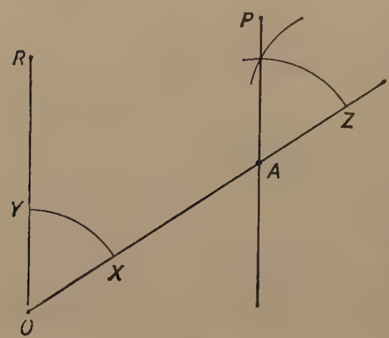


FIG. 6*e*

Draw the line through A parallel to OR.

To follow this instruction we draw three arcs.

(1) has centre O and any radius; it cuts OA at X and OR at Y.

(2) has centre A and the same radius as (1); it meets OA produced at Z.

(3) has centre Z and radius XY; it meets arc (2) at P. Then AP is the line through A parallel to OR (Fig. 6*e*).

The construction given uses only ruler and compasses. Parallels can also be drawn with the aid of a set-square.

To draw a line through P parallel to XY:

1. Place the set-square so that one of its sides lies along XY.
2. Place the ruler along a second side of the set-square.
3. Slide the set-square along the ruler until the first side of the set-square passes through P.
4. Draw the line along this first side. This will be the required line, passing through P and parallel to XY.

This construction is quicker and more convenient than the one previously given; but sometimes the use of set-square is forbidden in the instructions, and the compass construction must then be used.

If P is too far from XY to be reached in one movement of the set-square, alternate movements of ruler and set-square will enable us to reach P, but when the ruler is moved the set-square must be held firm and vice versa. If both ruler and set-square accidentally move together it is necessary to start again from the beginning.

In the first step of this construction the set-square should be placed to cover P if possible, as the construction can then be carried out with one movement of the set-square.

A set-square can also be used for drawing perpendiculars, if its use is not forbidden in the instructions. To draw a perpendicular from P to XY a good method is:

1. Place the set-square so that one of its perpendicular sides lies along XY and the set-square covers P.

2. Place the ruler along the oblique side and slide the set-square along the ruler until the other perpendicular side passes through P. The perpendicular from P to XY can then be drawn. This method is better than trying to make the set-square do two things at once, viz. have (1) one side lying along XY and (2) another passing through P.

Corresponding Angles

When two lines cross they form four angles. When two straight lines are crossed by a third, two groups of four angles are formed, one group at each crossing. A line crossing two lines is called a *transversal*. Our diagram, then, consists of two lines and a transversal. If we now move along the transversal, we shall reach first one group of four angles and then the other.

The members of each group can be distinguished by the words, near, far, left and right. Thus, when we reach the first group the four angles can be called near-left, near-right, far-left and far-right. When we come to the second group the four angles may be distinguished in the same way.

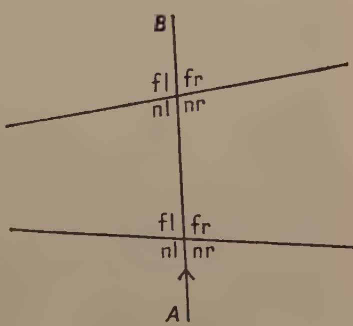


FIG. 7

The two angles which are each called near-left will be called *corresponding*; so will the two called near-right, far-left and far-right. In Fig. 7 we move along the transversal from A towards B. The letters n, f, r, l, are used to denote near, far, right, left, while corresponding angles are those which contain the same pair of letters.

Alternate Angles

In this same diagram, pairs of angles which *differ in both letters* (e.g. near-right of one group and far-left of the other) will be called *alternate*. Thus each angle of a group has a *corresponding* angle in the other group, and each angle of a group has an *alternate* angle in the other group. (Some writers reserve the word "alternate" for those angles formed by the part of the transversal between the two lines, but there seems no reason for this, and each angle of each group has both a corresponding and an alternate angle in the other group.)

When two parallel lines are crossed by a transversal, pairs of corresponding angles will be equal (*to one another*) *and pairs of alternate angles will be equal* (*to one another*). These statements are very useful in finding out facts about a figure which contains parallel lines. They are also useful in testing pairs of lines to determine whether they are parallel.

Vertically Opposite Angles

Returning for a moment to the group of four angles formed when two straight lines cross, each angle has two *adjacent* angles (i.e. sharing an arm with the angle). The fourth angle is said to be *vertically opposite* (to the one we are considering). Note that the words corresponding, alternate, adjacent, vertically opposite can never refer to a *single angle.*

The words should never be used unless it is clear that they refer to *two angles* both of which should be mentioned by name in the sentence we are writing.

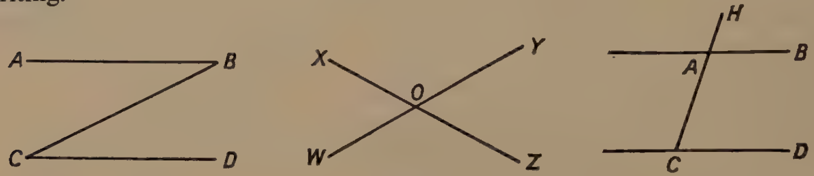


FIG. 8

Examples of correct use of the words (Fig. 8):
$A\hat{B}C$ and $B\hat{C}D$ are alternate. $X\hat{O}W = Y\hat{O}Z$ because they are vertically opposite. $H\hat{A}B = H\hat{C}D$ and they are corresponding. (Note that it is quite clear what is meant by the word "they.")

Examples of incorrect use of the words:
$A\hat{B}C$ is alternate. $X\hat{O}W$ is vertically opposite. $H\hat{A}B = H\hat{C}D$ and it is corresponding.

The same care should be taken with the words parallel and perpendicular, which should never be used unless it is clear that they refer to *two lines* both of which should be mentioned by name.

EXERCISE 1D

1. In Fig. 9*a* name:
(i) the angle alternate to a;
(ii) the angle corresponding to r;
(iii) the angle vertically opposite to s;
(iv) the angle alternate to q.

What kind (i.e. alternate, corresponding, or vertically opposite) are the following pairs: c, p; b, s; a, p; b, d; b, q; d, r.

2. In Fig. 9*b* name pairs of
(i) alternate;
(ii) corresponding;
(iii) vertically opposite angles.

3. In Fig. 9*c* CBD and EF are parallel. Write two equations (between pairs of angles) which follow from this fact.

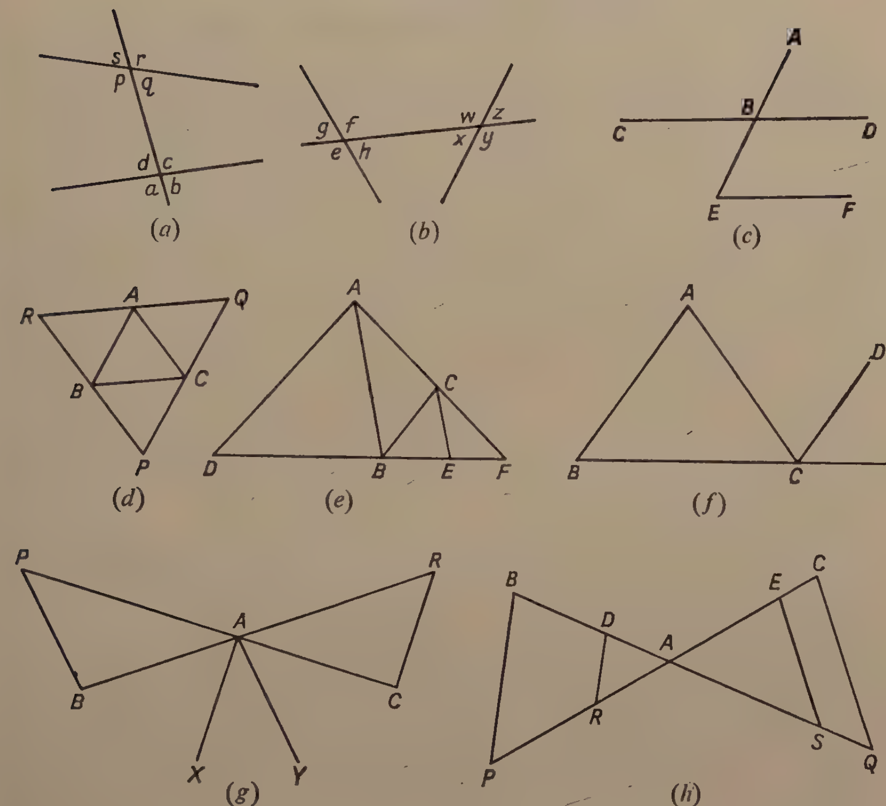


FIG. 9

4. Fig. 9*d* shows a triangle ABC with lines through the vertices parallel to the opposite sides. Since $QA \parallel CB$, we deduce that $Q\hat{A}C = A\hat{C}B$, as these angles are alternate. Omitting unnecessary words, we write $Q\hat{A}C = A\hat{C}B$ ($QA \parallel BC$: alt.).

Each acute angle in the diagram has two other angles equal to it. Write equations (similar to the sample above): (i) with subject $B\hat{A}C$; (ii) with subject $A\hat{B}R$; (iii) with subject $\hat{P}$. (Give two of each.)

5. In Fig. 9*e*, $AD \parallel CB$ and $AB \parallel CE$. Write two equations (as in Question 4) with subject $A\hat{B}C$. From your answers deduce that $D\hat{A}B = B\hat{C}E$.

6. In Fig. 9*f*, $A\hat{B}C = A\hat{C}B$ and $CD \parallel BA$. Deduce from these given facts that CA bisects $B\hat{C}D$.

7. In Fig. 9*g*, $\hat{B} = \hat{C}$, $AX \parallel RC$ and $AY \parallel PB$. Deduce from these given facts that $B\hat{A}Y = C\hat{A}X$.

8. In Fig. 9*h*, $\hat{B} = \hat{C}$, $DR \parallel BP$ and $ES \parallel CQ$. Use these given facts to deduce that $R\hat{D}A = S\hat{E}A$.

The Angles of a Triangle

If we plot three points, A, B and C on our paper, and we draw the straight lines from B to C, from C to A and from A to B, we have formed a triangle. (If we find that one straight line will pass through A, B and C, we start again and choose our three points differently.) In a triangle are three angles. If we produce one of the sides (e.g. continue the straight line BC beyond C) we form a fourth angle. Such an angle is called an *exterior angle* of the triangle. The angles in the original triangle will be called *interior angles* whenever it is necessary.

Thus any triangle has three interior angles and we may give it six exterior angles.

There are some useful and well-known facts about these angles. Here are three:

An exterior angle at a vertex is equal to the sum of the interior angles at the other two vertices.

The sum of the interior angles is two right angles.

The sum of three exterior angles, one from each vertex, is four right angles.

We can prove these statements to be true (of any triangle).

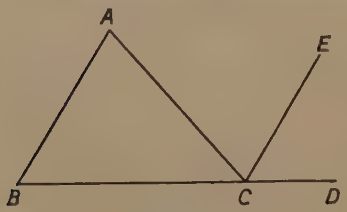


FIG. 10

We are given a triangle with one side produced. We are asked to prove (i.e. to give reasons why) that the exterior angle is equal to the sum of the two interior angles at the other two vertices. We begin by lettering the diagram so that we can talk more easily about it. Any letters may be chosen provided the same letter is not used for two different points. We announce our choice of letters in the first sentence (Fig. 10).

Given: A triangle ABC with BC produced to D.

To prove: $A\hat{C}D = \hat{A} + \hat{B}$. (Note that there is only one angle in the diagram with vertex A, and this angle may be named by a single letter. But there are several at C.)

Construction: CE is parallel to BA. (The point E is going to help with our reasons, but it is not part of the original diagram. Under the heading "Construction" we name any such points.)

Proof: $\hat{A} = A\hat{C}E$ because they are alternate and BA is parallel to CE; $\hat{B} = E\hat{C}D$ because they are corresponding and BA is parallel to CE.

Hence $\hat{A} + \hat{B} = A\hat{C}E + E\hat{C}D$ and $A\hat{C}E + E\hat{C}D = A\hat{C}D$.

Hence $A\hat{C}D = \hat{A} + \hat{B}$.

The argument can be carried a stage further, for starting with $\hat{A} + \hat{B} = A\hat{C}D$, we can say $\hat{A} + \hat{B} + B\hat{C}A = A\hat{C}D + B\hat{C}A$, and the left side of this equation is the sum of the angles of our triangle, and the right side is equal to the straight angle BCD, so we have proved that:

The sum of the angles of a triangle is two right-angles.

These two facts, which are true of any triangle, enable us to calculate the angle between two lines when we know the angles they make with a third line.

In the following examples we must calculate the angle between the two fully-drawn lines when we are given the angles they make with the dotted line.

Example 1: Given $\hat{A} = 57°$ and $\hat{D} = 49°$, calculate $A\hat{B}C$. (Fig. 11*a*).

(Calculation)

$A\hat{B}C = \hat{A} + \hat{D}$ (exterior angle of a triangle)

$= 57° + 49° = 106°$

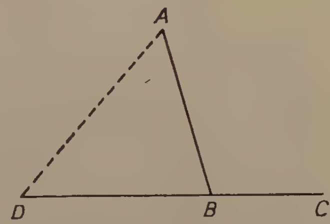


FIG. 11*a*

Example 1a: Given $\hat{A} = x°$ and $\hat{D} = y°$, calculate $A\hat{B}C$.

(Calculation)

$A\hat{B}C = \hat{A} + \hat{D}$ (ext. ang. $\triangle$)

$= x° + y° = (x + y)°$

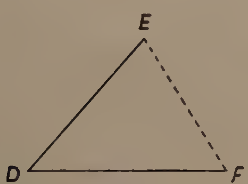


FIG. 11*b*

Example 2: Given $\hat{E} = 72°$, $\hat{F} = 58°$, calculate $\hat{D}$.

(Calculation) $\hat{D} + \hat{E} + \hat{F} = 2$ right-angles (angle-sum of triangle)

(Putting in what we know)

$\hat{D} + 72° + 58° = 180°$

(Solving the equation)

$\hat{D} = 180° - (72° + 58°)$

$= 180° - 130° = 50°$

Example 2a: Given $\hat{E} = x°$ and $\hat{F} = y°$, calculate $\hat{D}$.

(Calculation) $\hat{D} + \hat{E} + \hat{F} = 2$ right-angles (angle-sum of triangle)

(Putting in what we know)

$\hat{D} + x° + y° = 180°$

(Solving the equation) $\hat{D} = 180° - (x + y)°$

Example 3: Given $L\hat{Y}Z = 153°$ and $M\hat{Z}Y = 162°$, calculate $K\hat{X}Y°$. (Fig. 11*c*)

(Calculation) $X\hat{Y}Z = 180° - L\hat{Y}Z$

$= 180° - 153°$

$= 27°$;

$X\hat{Z}Y = 180° - M\hat{Z}Y$

$= 180° - 162°$

$= 18°$.

$K\hat{X}Y = XYZ + X\hat{Z}Y$

$= 27° + 18°$

$= 45°$

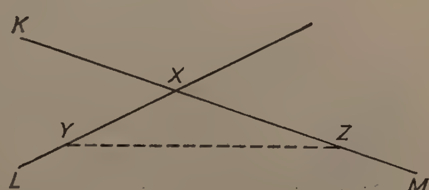


FIG. 11*c*

Example 3a: Given $L\hat{Y}Z = x°$ and $M\hat{Z}Y = y°$, calculate $K\hat{X}Y$.

(Calculation) $X\hat{Y}Z = 180° - L\hat{Y}Z = (180 - x)°$;

$X\hat{Z}Y = 180° - M\hat{Z}Y = (180 - y)°$.

$K\hat{X}Y = X\hat{Y}Z + X\hat{Z}Y = (180 - x)° + (180 - y)°$

$= (180 - x + 180 - y)° = (360 - x - y)°$

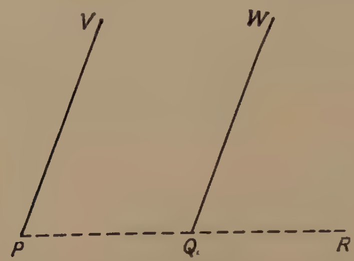


FIG. 11*d*

Example 4: Given $\hat{P} = 68°$ and $W\hat{Q}R = 69°$, calculate the angle between PV and QW (Fig. 11*d*).

Note that these two lines would meet off the paper, but we can nevertheless calculate the angle at which they would meet.

(Calculation) Let PV and QW meet at X. (Name the point so that we can talk about it.) Then

$P\hat{X}Q + X\hat{P}Q = X\hat{Q}R$ (external angle $\triangle$)

$P\hat{X}Q + 68° = 69°$

so that $P\hat{X}Q = 69° - 68° = 1°$

i.e. the angle between PV and QW is $1°$.

Note that in these examples we are able to calculate the angle between two lines if we know the angle which each makes with a third line. This method is of very great use in calculating angles in complicated diagrams.

The Four Processes of Geometry

When we are given a figure we may be asked to apply one or more of four processes to the figure.

1. We may be asked to put new points into the figure given their locations. This process will be called CONSTRUCTION.
2. We may be asked to use ruler and protractor to measure lengths and angles in the figure. This process will be called MENSURATION.
3. We may be asked to calculate lengths and angles in the figure without the use of ruler or protractor. This process will be called CALCULATION.
4. We may be asked to prove certain facts about the figure. This process will be called DEDUCTION.

In Construction and Mensuration problems it is necessary to draw as accurate a figure as possible, and this requires a skilful use of the geometrical instruments.

The constructions on page 298 provide a test of accuracy. If the circle through A, B and C is drawn, following the instructions on that page, it requires a certain amount of skill to obtain a circle which passes through all three points, and lack of skill will result in a circle which misses one or more of the points by a visible amount.

Results of mensuration problems can be calculated (by methods which we have not so far met) and given to a greater degree of accuracy than we can obtain with ruler and protractor. These calculated results when given will show how skilful we are at measurement.

In Calculation and Deduction problems there is no need for an accurate diagram. Even a freehand diagram will do and there is no need for it to be accurate. The figures for the calculation problems on page 305 were drawn without using a protractor and it is unlikely that any of the angles in these figures has the size it is supposed to have.

EXERCISE 1E

Construction and Mensuration Problems

1. Plot A and B 3 in apart. Plot the point C such that $AC = 2.4$ in $BC = 3$ in. Measure the angles A, B and C.

2. Plot A, B and C such that $AB = 4{\cdot}5$ cm, $A\hat{B}C = 1$ right-angle and $BC = 6$ cm. Measure AC and the angles A and C of the triangle ABC.

3. Plot O and A, 3 in. apart. Plot B such that $OB = 2$ in. and $A\hat{O}B = 57°$. Plot C such that AC is parallel to OB and BC is parallel to OA. (The figure $OACB$ is called a *parallelogram* because its opposite sides are parallel.) Measure the lengths of AC and BC.

4. Plot O and R, 7·5 cm apart. Plot A from the location $R\hat{O}A = 68°$, $OR = 5$ cm. Plot M from the facts: (1) AM is perpendicular to OR; (2) M lies in OR. Measure AM and OM. (AM is called the *perpendicular distance from A to OR*, or simply the *distance from A to OR*.)

5. In the diagram of No. 4 measure the distance from O to AR.

6. A ship sails 48 km on a bearing 220°. It then sails 32 km on a bearing 020° (i.e. 20°; bearings should always have three figures). Measure the bearing it must now sail on to return to its starting point. Take a scale of 2 cm to represent 10 km, name the starting point O, the first stop A and the second stop B.

Angle Calculations

When two lines cross they form four angles. If we know one of these four angles we can calculate the other three. Using $x°$ to stand for the size of the angle we know, the sizes of the other three will be $x°$ (the vertically opposite angle) and $(180 - x)°$ (each adjacent angle). If a third line is now drawn it will form four angles at each intersection with the other two.

Thus we have three groups, each of four angles, and if we know the size of a member of one group and the size of a member of another group, we can calculate the size of any member of the third group. The calculation of an angle in a diagram will usually require a geometrical fact, and this fact should always be clearly stated in the calculation, and the angles concerned should be referred to by name.

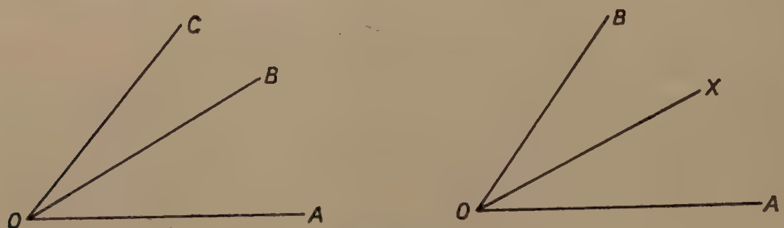


FIG. 12

In this very simple example in which $A\hat{O}B = 30°$ and $B\hat{O}C = 20°$, it is easy to see that $A\hat{O}C = 50°$ and the arithmetic required is of the simplest type. But we are practising Geometry, and the geometrical fact required is $A\hat{O}C = A\hat{O}B + B\hat{O}C$ (which needs no reason) and this fact should be the first step of the calculation. The arithmetic part (adding 20 to 30) may be done in the head, and the calculation, properly done, will read:

$$\begin{aligned} A\hat{O}C &= A\hat{O}B + B\hat{O}C \\ &= 30° + 20° \\ &= 50° \end{aligned}$$

It is good practice to begin each new calculation with a geometrical fact of this kind, giving a reason if one is necessary. The facts which we shall most often use are:

Adjacent angles are *supplementary* (i.e. their sum is two right-angles).

The sum of two adjacent angles is two right-angles (adj. $\angle$s).

An exterior angle of a triangle is equal to the sum of the other two interior angles (ext. $\angle\triangle$).

The sum of the angles of a triangle is two right-angles (angle sum $\triangle$).

If two sides of a triangle are equal, then these sides make equal angles with the third side. A triangle with two equal sides is called ISOSCELES, and if it has three equal sides it is called EQUILATERAL.

Two angles are said to be *supplementary* if their sum is two right-angles.

Each is said to be the *supplement* of the other. To calculate the supplement of an angle we subtract the angle from two right-angles, or 180°. To prove two angles supplementary we calculate their sum which we then compare with 180°. The abbreviations in brackets show how to refer to these facts when using them in calculations or deductions.

A line is said to *bisect an angle* when it passes through the vertex of the angle and it makes equal angles with the two arms (of the angle). If, in the figure, we know that $A\hat{O}X = X\hat{O}B$, we may say that OX bisects $A\hat{O}B$, or

OX is the bisector of $A\hat{O}B$.

Algebra should be freely used in calculations and if an angle whose size is to be calculated is taken as $x°$, the use of one or more of the foregoing facts should enable us to form an equation from which the value of x may be found.

Example: $A\hat{O}B$ is a straight angle; $A\hat{O}C = 102°$; OX bisects $B\hat{O}C$. Calculate $B\hat{O}X$.

(Yes: the answer is 39° but we are learning how to do *geometrical* calculations!)

$$\begin{aligned} \text{Let } B\hat{O}X &= x°. \\ \text{Then } B\hat{O}C &= 2x° \\ B\hat{O}C + A\hat{O}C &= 180° \\ 2x° + 102° &= 180° \\ 2x &= 78 \\ x &= 39 \\ \textit{Answer: } B\hat{O}X &= 39° \end{aligned}$$

CHAPTER 2

SHAPE AND SIMILARITY

Given any two points, O and R, plot A from $\dfrac{OA}{OR} = \frac{2}{3}$; $R\hat{O}A = 40°$.

To follow these instructions we
(1) plot O anywhere;
(2) draw a line in any direction from O;
(3) make 3 equal steps with compasses along this line from O, marking the end of the third step R;
(4) draw the line through O making 40° with OR; step off along this line 2 steps with the compasses, marking the end of the second step A.

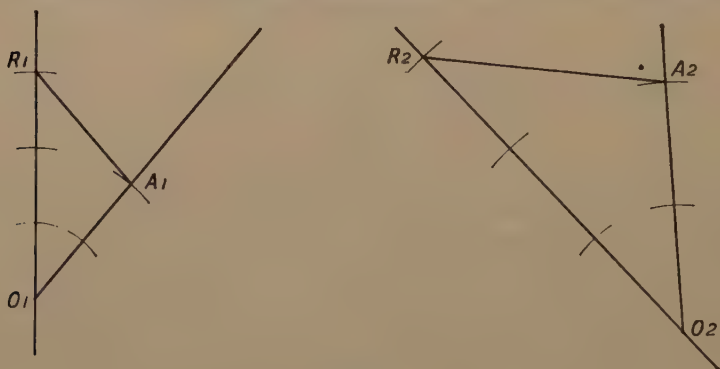


FIG. 1

Fig. 1 shows these instructions followed twice, with two positions for O (O_1 and O_2), two for R (R_1 and R_2), leading to two positions for A (A_1 and A_2). (A larger step with the compasses was used in the second figure.)

You should now experiment by drawing a pair of triangles, using other numbers in place of the 2, 3 and 40 of the example above. We now have a pair of triangles related by the two facts:

$$\frac{O_1A_1}{O_1R_1} = \frac{O_2A_2}{O_2R_2}$$

and $\hat{O}_1 = \hat{O}_2$.

We are not surprised to find that they are also related by the following facts:

$$\hat{R}_1 = \hat{R}_2, \quad \hat{A}_1 = \hat{A}_2,$$

$$\frac{A_1R_1}{A_1O_1} = \frac{A_2R_2}{A_2O_2},$$
$$\frac{A_1R_1}{O_1R_1} = \frac{A_2R_2}{O_2R_2}$$

Thus the two triangles are related by these six facts, of which the first two are the result of our construction and the other four follow from the first two. The first two may be called the *given* facts and the other four the *deduced* facts.

Similar Triangles

The word SIMILAR is used for triangles related by six facts in this way (i.e. agreeing about three angles and three ratios) and we may combine these six facts into the single statement

$O_1A_1R_1$ is similar to $O_2A_2R_2$, or, using symbols,
$O_1A_1R_1 \;|||\; O_2A_2R_2$, or
the triangles $O_1A_1R_1$ and $O_2A_2R_2$ are similar.

The sign $|||$ is read as "is similar to" and it should be used as a phrase and not as the adjective "similar."

The values of the three angles are $\hat{O} = 40°$ (by construction), $\hat{A} = 98{\cdot}8°$ and $\hat{R} = 41{\cdot}2°$ (by a calculation by trigonometry).

The values of the three ratios are $\dfrac{OA}{OR} = \frac{2}{3}$ (by construction), $\dfrac{AR}{AO} = {\cdot}976$, $\dfrac{AR}{OR} = {\cdot}650$ (by calculation by trigonometry).

In giving these six values two things should be noted:

1. That there are no suffixes since the values apply to either triangle.
2. The calculated results are given in decimals. This implies that they are approximations, whereas the values of $\hat{O}$ and $\dfrac{OA}{OR}$ are known exactly. Approximate values should always be given in decimals.

These six values may be called the *shape-items* of the triangle, and the complete six constitute the *shape* of the triangle. We may thus think of the triangles as having "the same shape," meaning that their six shape-items agree, but we shall continue to use the word similar in referring to such triangles.

In our experiment we made the triangles agree about two of their shape-items and we found that they automatically agreed about the other four. The two shape-items which we chose were an angle and the ratio of the arms of that angle. We can also experiment by making triangles agree about other pairs of shape-items.

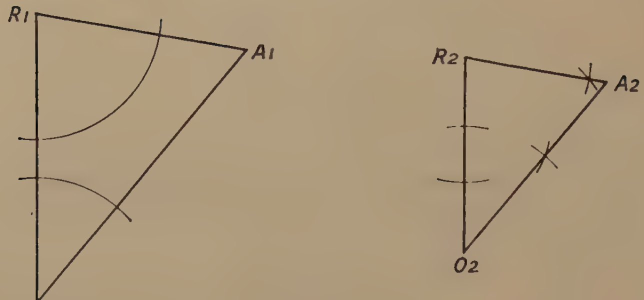


FIG. 2

We make a plot from the relations $R_1\hat{O}_1A_1 = R_2\hat{O}_2A_2$ and $A_1\hat{R}_1O_1 = A_2\hat{R}_2O_2$. We choose O_1, O_2, R_1, R_2, A_1 at random and A_2 can now be plotted (Fig. 2). (It is unnecessary to use the protractor to make the angles equal; the construction can be done with compasses.) Again having made the triangles agree about two shape-items, we find that they automatically agree about the other four.

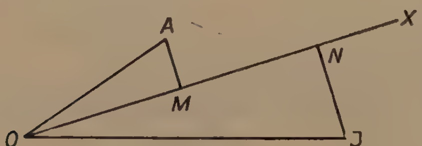


FIG. 3

This case, in which the triangles agree about two angles, is the most common one in working with similar triangles, and the other four shape-items may be taken as agreeing at once. In solving problems based on similar triangles the argument is usually written as follows (the remarks in italics being explanatory, and not part of the written work):

Given: OX bisects $A\hat{O}B$; AM and BN are perpendicular to OX (Fig. 3).

To prove: $\dfrac{OA}{AM} = \dfrac{OB}{BN}$

Proof: In the triangles OAM and OBN (*this names the triangles we consider*) $A\hat{O}M = B\hat{O}N$ (given)

$A\hat{M}O = B\hat{N}O$ (right angles, given)

We have now established that the two triangles agree about two shape-items, so they agree about all shape-items. We write, shortly,

$$\therefore \triangle AOM \,|||\, \triangle BON$$

This process is referred to as "proving the triangles similar" and we now have the right to use any of the other four equal shape-items.

So we say $\dfrac{OA}{AM} = \dfrac{OB}{BN}$ *which is what we set out to prove.*

We have seen so far that two triangles are similar if they agree about two angles, or if they agree about the angle and the ratio at one of the vertices.

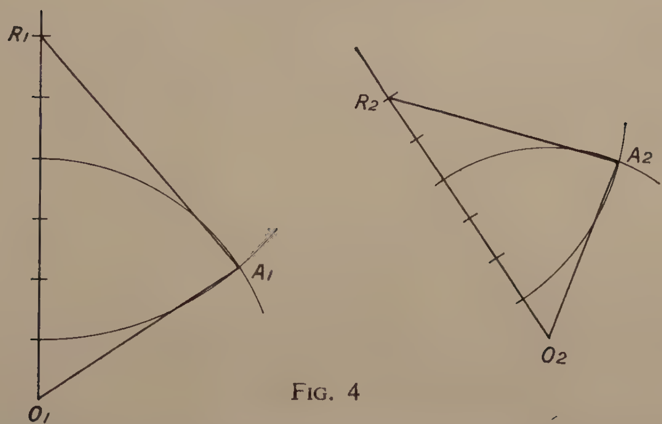


FIG. 4

We find also that triangles are similar if they agree about two ratios, e.g. Fig. 4 shows O, R and A plotted (twice) from the location $\dfrac{OA}{OR} = \frac{2}{3}$, $\dfrac{RA}{OR} = \frac{5}{6}$. (Six steps have been marked off from O to R. One of the arcs has centre O radius 4 steps, the other has centre R radius 5 steps.)

In this figure *we have constructed:*

$$\frac{O_1A_1}{O_1R_1} = \frac{O_2A_2}{O_2R_2} \textit{ and } \frac{R_1A_1}{O_1R_1} = \frac{R_2A_2}{O_2R_2} \textit{ so we can deduce}$$

$$\therefore\ O_1R_1A_1 \ ||| \ O_2R_2A_2$$

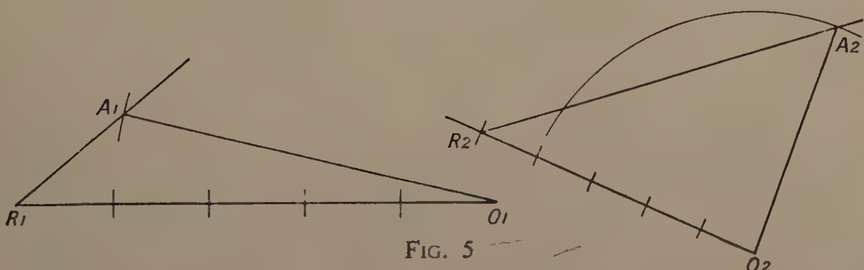


FIG. 5

There is a case of failure, for it is possible to draw two triangles which agree about two shape-items but do not agree about all. The failure occurs when one of the items is an angle and the other is a ratio, but the items are not at the same vertex. e.g. to plot A from the location $\dfrac{AO}{OR} = \dfrac{4}{5}$, $O\hat{R}A = 40°$, we would plot O, then R 5 steps away, then the line through R making 40° with RO, and finally an arc with centre O and radius 4 steps. Fig. 5 shows two triangles drawn from these instructions. They agree about two shape-items, for $\dfrac{O_1A_1}{O_1R_1} = \dfrac{O_2A_2}{O_2R_2}$ and $O_1\hat{R}_1A_1 = O_2\hat{R}_2A_2$ by construction and the other

four shape-items are all seen to disagree. The reason for the failure is that the agreeing ratio is at the vertex O and the agreeing angle at the vertex R. This case must be excluded from our list, and the complete result may be stated as follows:

If two triangles agree about

two angles, or
two ratios, or
the angle and ratio at one vertex,

then the triangles are similar.

Similar triangles are of very frequent occurrence in geometry. We are so familiar with them that we use them unconsciously. Three points on a map, for example, form a triangle similar to that formed by the three places which they represent. Three points in a photograph form a triangle similar to that formed by the same three points in an enlargement. We are also unconsciously using the idea of similar triangles when we say that two objects have the same shape.

Our concern with similar triangles in geometry is to recognize them whenever it is necessary, and to use them in calculation and deduction. Fig. 6 shows a triangle ABC and several similar triangles. $A_1B_1C_1$ is related to ABC by having its sides drawn parallel (to those of ABC).

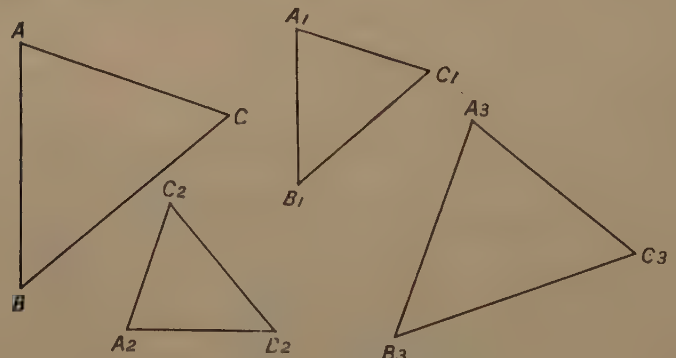


FIG. 6

$A_2B_2C_2$ is related to ABC by having its sides drawn perpendicular (to those of ABC), i.e. AB and A_2B_2 would form a right-angle if made to meet. In the triangle $A_3B_3C_3$ the bearing of each side is 20° more than the bearing of the (corresponding) side of ABC, e.g. if the bearing of AB is $x°$, then the bearing of A_3B_3 is $(x + 20)°$.

These three triangles are similar to the triangle ABC and they are also similar to each other.

Working with Ratios

The expression $\dfrac{XY}{AB}$ is used to denote the number of times AB can be stepped along XY. When we write $\dfrac{XY}{AB} = 13$ we mean that 13 steps equal to AB would take us from X to Y; when we write $\dfrac{XY}{AB} = \frac{13}{5}$ we mean that a step contained 5 times in AB would be contained 13 times in XY; when we write $\dfrac{XY}{AB} = 2{\cdot}6$ we normally mean that AB is contained *approximately* 2·6 "times" in XY (and also that 2·6 is a better approximation than 2·5 or 2·7); when we write $\dfrac{XY}{AB} = p$ we mean that p is to be taken as the number of times (whatever it may be) which AB can be stepped along XY. Note that in each case the value of the ratio $\dfrac{XY}{AB}$ is a number.

If AB can be stepped p times along XY, and XY can be stepped q times along CD, then AB can be stepped pq times along CD. This statement can be seen to be true, by actual counting, when p and q are specified whole numbers. We shall take it as being true when p and q are any kind of number.

Using symbols, the result reads:

If $\dfrac{XY}{AB} = p$ and $\dfrac{CD}{XY} = q$, then $\dfrac{CD}{AB} = pq$, or, without mentioning p and q, $\dfrac{CD}{AB} = \dfrac{CD}{XY}\cdot\dfrac{XY}{AB}$ (where the full-stop means "multiplied by").

We note that the XY "cancels" just as equal numerators and denominators cancel when arithmetical fractions are multiplied together, but this cannot be given as a *reason* for the statement that

$$\frac{CD}{AB} = \frac{CD}{XY}\cdot\frac{XY}{AB}$$

EXERCISE 2A

Construct statements of this kind using the ratios:

1. $\dfrac{PQ}{AB}, \dfrac{GH}{AB}, \dfrac{PQ}{GH}$
2. $\dfrac{HK}{RS}, \dfrac{HK}{LM}, \dfrac{LM}{RS}$
3. $\dfrac{AB}{BC}, \dfrac{BC}{CD}, \dfrac{AB}{CD}$
4. $\dfrac{WX}{XY}, \dfrac{WX}{YZ}, \dfrac{XY}{YZ}$
5. Given $\dfrac{PQ}{GH} = 3$, $\dfrac{GH}{AB} = 2$, calculate $\dfrac{PQ}{AB}$, using the result of Question 1.

6. Using the result of Question No. 2, calculate

$$\frac{HK}{LM}, \text{ given } \frac{HK}{RS} = 2, \frac{LM}{RS} = 5.$$

7. Using the result of Question No. 3, find $\frac{AB}{CD}$ given $\frac{AB}{BC} = x, \frac{BC}{CD} = y.$

8. Using the result of Question No. 4, find $\frac{WX}{XY}$ given $\frac{WX}{YZ} = p, \frac{XY}{YZ} = q.$

In the following examples begin each step with a statement about ratios as in the answers to Questions 1-4.

Example: Given $\frac{OA}{OB} = 1{\cdot}3, \frac{OB}{OC} = 4$, calculate $\frac{OA}{OC}$

Statement:	$\frac{OA}{OC} = \frac{OA}{OB}.\frac{OB}{OC}$
Put in what we know:	$= 1{\cdot}3 \times 4$
Work out:	$= 5{\cdot}2$
Answer:	$\frac{OA}{OC} = 5{\cdot}2$

9. Given $\frac{AB}{BC} = 3{\cdot}7, \frac{BC}{CD} = 1{\cdot}2$, calculate $\frac{AB}{CD}$

10. Given $\frac{AB}{BC} = 3, \frac{BC}{CD} = 2, \frac{CD}{DE} = 1{\cdot}6$, calculate (i) $\frac{AB}{CD}$; (ii) $\frac{AB}{DE}$

11. Given $\frac{AB}{GH} = 4, \frac{PQ}{XY} = 5, \frac{PQ}{GH} = 2$, calculate $\frac{AB}{XY}$

Symbols as Units of Distance

The symbol $\frac{AB}{OR}$ represents the number of steps, equal to OR, contained in AB. Now if we agree to keep to the same step in any piece of work, we may also agree to omit the denominator of the expression and write it simply as AB.

We shall take the two-letter symbol AB, then, as meaning the number of steps (of agreed size) from A to B, and when several of these two-letter symbols occur in the same piece of work, we agree that the same unit is to be used in all cases.

These two-letter symbols will thus represent numbers, and can be combined like numbers.

Thus $AB + BC$ is the result of adding two numbers, and $AB.BC$ the result of multiplying them together.

$AB^2 - BC^2$ is the result of squaring two numbers and taking one result from the other—and so on.

Example: If $AB = 3$ and $BC = 4$:

$$AB.BC = 12$$
$$AB + BC = 7$$
$$AB^2 + BC^2 = 9 + 16$$
$$= 25$$
$$(AB + BC)^2 = 7^2 = 49$$
$$\frac{AB}{BC} = \tfrac{3}{4}$$
$$\text{and } \frac{AB}{BC} + \frac{BC}{AB} = \tfrac{3}{4} + \tfrac{4}{3}$$
$$= \tfrac{25}{12}.$$

Most important of all, $\frac{AB}{BC}$ still has its original meaning, the number of steps, equal to BC, contained in AB. We note also that the step can be specified by giving it the value 1, and if we wish to choose, say, OR as a step, we write simply $OR = 1$.

When we write $AB = AC$ we say that the distances from A to B and from A to C contain equal numbers of steps, so that the lengths of AB and AC are equal. But when we write $AB = 5$, we do not know how far A and B are apart until we have been told the unit to be used.

EXERCISE 2B

1. Given $AB = 3$, $BC = 4$, $XY = 2\frac{1}{2}$ and $XZ = 1{\cdot}41$, calculate:

(a) $AB.XY + BC.XY$;

(b) $BC^2 + XZ^2$;

(c) $BC.XY - AB.XZ$;

(d) $\dfrac{BC.XY}{BC + 2AB}$;

(e) $BC^2 - (AB^2 + XY^2)$

2. Solve the equations:

(a) $3.AB + 10 = 5.AB$;

(b) $\dfrac{AB}{9} = \dfrac{4}{AB}$;

(c) $\dfrac{XY + 5}{XY - 5} = 2$;

(d) $(GH + 4)(GH - 3) = GH^2$

3. Factorize:

(a) $AB.XY + AB.XZ$;

(b) $PQ^2 - PQ.PR$;

(c) $AB.(XY + XZ) - CD.(XY + XZ)$;

(d) $AB^2 - AB$.

4. In Fig. 7, $A\hat{B}C = \hat{D}$ (so that $ABC \,|||\, ADB$), $AC = 4$, $CD = 5$. Calculate AB.

(Use the shape-items $\dfrac{AB}{AC}$ and $\dfrac{AD}{AB}$ of the similar triangles.)

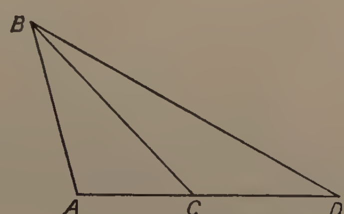


FIG. 7

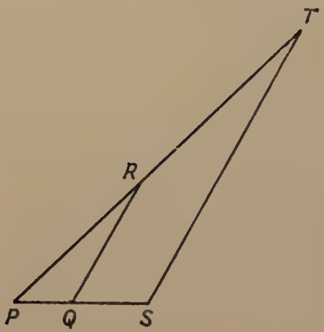


FIG. 8

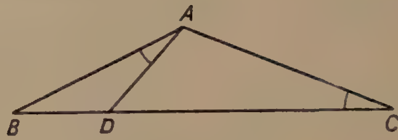


FIG. 9a

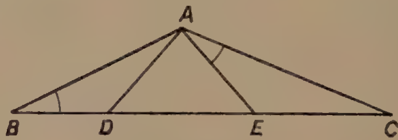


FIG. 9b

5. In Fig. 8, $PQ = 3$, $QS = 4$, $PR = 9$, $QR = 7$ and $QR \parallel ST$. Calculate PT and ST.

6. In Fig. 9a, $AB = 5$, $AC = 6$ and $BC = 10$. Also, $B\hat{A}D = \hat{C}$. By using the similar triangles BDA and BAC, calculate AD and BD.

7. AE is added, making $E\hat{A}C = \hat{B}$ (Fig. 9b). Calculate DE.

8. In Fig. 10, $Q\hat{P}S = \hat{R}$ and $T\hat{P}R = \hat{Q}$. Prove
(a) $PQ^2 = QR.QS$;
(b) $PR^2 = QR.RT$;
(c) $QR^2 - PQ^2 - PR^2 = QR.ST$

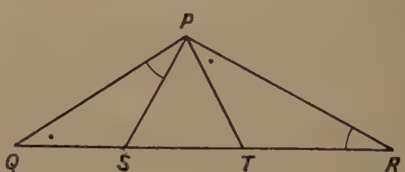


FIG. 10

Pythagoras' Theorem

In Fig. 11 $B\hat{A}C$ is a right-angle and AD is perpendicular to BC.
In the triangles ABD and CBA,

$$\hat{B} = \hat{B} \text{ and } A\hat{D}B = C\hat{A}B \ (rt. \angle\, s)$$

$$\therefore ABD \,|||\, CBA \therefore \frac{BA}{BD} = \frac{BC}{BA} \therefore BA^2 = BC.BD$$

Similarly from the triangles ACD and BCA, $CA^2 = CB.CD$.

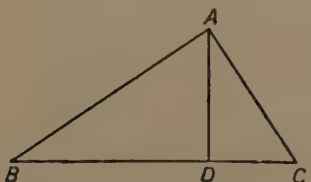


FIG. 11

Hence $BA^2 + CA^2 = BC.BD + BC.CD = BC.(BD + DC) = BC.BC = BC^2$, i.e. $BA^2 + CA^2 = BC^2$.

This theorem is the best-known theorem in geometry and was first proved by Pythagoras. It is true of any right-angled triangle, and it enables us to calculate the third side of a

right-angled triangle when we are given two of the sides. The longest side of a right-angled triangle (i.e. the one facing the right-angle) is called the *hypotenuse* of the triangle, and the theorem is usually stated as:

In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

When Pythagoras announced his theorem, the "square on the hypotenuse" meant the area of the square drawn with the hypotenuse as one side, and similarly for the squares on the other two sides. Our symbols represent numbers, and all that we have shown is that the sum of the squares of two numbers is equal to the square of a third. But BC is the number of units in BC, and if a square is drawn with BC as one side, and then this square is divided into unit squares (i.e. squares with our unit as one side), the number of these unit squares is obtained by squaring the number of units in BC.

Thus the symbol BC^2 means either the number of units in BC multiplied by itself, or the number of square units in the square drawn on BC. Thus our statement of the theorem and Pythagoras' statement come to the same thing.

EXERCISE 2C

Exercises on Pythagoras' Theorem.

1. In $\triangle ABC$, $AB = 7$, $AC = 24$ and $\hat{A}$ is a right-angle. Calculate BC. The calculation should be clearly set out as follows:

$$\begin{aligned} BC^2 &= AB^2 + AC^2 \text{ (Pythagoras)} \\ &= 7^2 + 24^2 \\ &= 49 + 576 \\ &= 625. \qquad BC = 25. \end{aligned}$$

2. In $\triangle DEF$, $DF = 5$, $EF = 4$ and $\hat{E}$ is a right-angle. Calculate DE.

3. In $\triangle KLM$, $KL = KM = 13$ and $LM = 10$. KN is the perpendicular from K to LM. Calculate KN. (KN may be taken as bisecting LM.)

4. A ship sails 15 miles East and then 8 miles North. How far is it from its starting point? (Draw and label a figure showing the three points. It need not be drawn accurately.)

5. Calculate the diagonal of a rectangle whose sides are 8 m and 6 m.

Answers Which do not "Work Out"

In the foregoing exercises the numbers have been chosen so that the answers "work out." This will not always be the case.

Example: In $\triangle PQR$, $PQ = 1$, $PR = 1$ and $\hat{P}$ is a right-angle.

Calulate $QR^2 = PQ^2 + PR^2$ (Pyth.)

$$= 1 + 1 = 2$$

The only *exact* solution to this equation is $QR = \sqrt{2}$.
If the question is now asked: "What does $\sqrt{2}$ mean?", a good answer is: "I don't quite know, but its square is 2." It may be used like any other mathematical symbol.

Example:

$$\sqrt{2} + \sqrt{2} = 2.\sqrt{2},$$
$$5.\sqrt{2} + 3.\sqrt{2} = 8.\sqrt{2}$$
$$\text{and } 7.\sqrt{2} - 6.\sqrt{2} = \sqrt{2}$$
$$\text{in the same way as } a + a = 2a$$
$$5a + 3a = 8a$$
$$\text{and } 7a - 6a = a,$$

and just as there is no simpler way of writing $3a + 5$, so there is no simpler way of writing $3.\sqrt{2} + 5$. But there is a simpler way of writing $\sqrt{2}.\sqrt{2}$ namely, 2, whereas there is no simpler way of writing a^2 or $a.a$.

A number of this kind is called a SURD and it is impossible to write its exact value without using a root sign. There are many approximations to its value, e.g. 1·4142 and $\frac{99}{70}$, but none of these is equal to $\sqrt{2}$. The square of the first of these approximations is 1·99996164, and of the second $2\frac{1}{4900}$, both being very close to 2.

Surds will appear often in calculations using Pythagoras' Theorem. In these cases, the exact value, in surd form, should be given, and followed by an approximation correct to three—or four—significant figures.

EXERCISE 2D

1. Calculate the diagonal of a rectangle whose sides are 8 m and 5 m.

2. In Fig. 12, $ST = 17$, $SV = 10$, $SU = 8$ and SU is perpendicular to TV. Calculate TV.

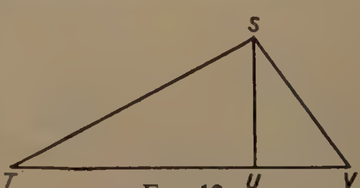


FIG. 12

3. NPQ is an equilateral triangle and NP is 2. Calculate the perpendicular from any vertex to the opposite side.

4. In Fig. 13, $OA = AB = BC = CD = 1$ and $O\hat{A}B = O\hat{B}C = O\hat{C}D = 1$ right-angle. Calculate OD. (The exact value should be given; no approximations are needed in the calculation.)

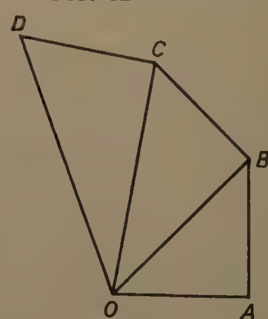


FIG. 13

5. Use the fact that $9 + 1 = 10$ to construct a line of length $\sqrt{10}$ in.

6. Two vertical posts are 11 m and 16 m high respectively and they are 12 m apart. Calculate the distance between the tops of the posts.

7. Two ships set out from the same point. The first travels 8 miles east and then 12 miles north. The second travels 4 miles west and then 7 miles north.

(i) How much farther north is the first ship than the second?
(ii) How much farther east is the first ship than the second?
(iii) How far are the ships apart?

8. Two circles of radius 17 cm and 25 cm respectively meet at two points which are 30 cm apart. Calculate the distance between the centres of the circles. (The line joining the centres, A and B, meets the line joining the meeting points, P and Q, at right-angles.)

9. $ABCD$ is a square whose diagonal, AC, is 4 inches long. Calculate the length of a side of the square.

10. The foot of a ladder is 6 ft from the bottom of a wall 14 ft high and the ladder rests against the top of the wall. Given that the length of the ladder is 21 ft, calculate the length of ladder which projects over the top of the wall. Give the answer correct to three significant figures.

CHAPTER 3

AREA: AREA OF A TRIANGLE: CONSTANT AREA LOCUS

EACH OF these figures is obtained by following a path starting at A and ending at A. The word *closed* is used to mean "starting and ending at the same point." We are not concerned with those which cross themselves at some point, as in Figs. 2 and 5. In the others, the path is said to enclose an area, and the path is called the perimeter (or boundary) of the area. If part (or the whole) of the perimeter is curved, as in Figs. 1, 2 and 3, the area is too difficult to measure at this stage. So we shall be concerned only with areas whose perimeters are composed of straight lines, i.e. polygons.

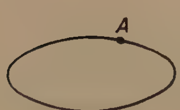


FIG. 1

FIG. 2

FIG. 3

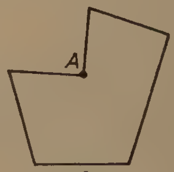


FIG. 4

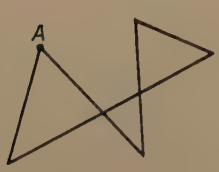


FIG. 5

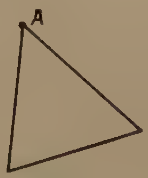


FIG. 6

FIG. 7

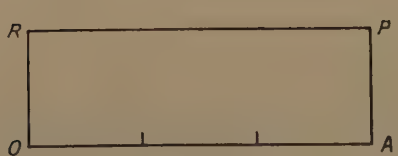


FIG. 8

To measure an area we first need a unit, or standard, area. All our diagrams are related to a unit step and it is convenient to take the square on this unit step as the unit area. An area can then be measured by the number of such squares required to cover it.

Fig. 8 shows a rectangle $OAPR$ in which OR is the unit step and $OA = 3$. The area of this rectangle is 3 units, the same as the number of steps in OA. Fig. 9 shows a rectangle $OAPB$ in which $OA = 4$ and $OB = 3$. (The unit step is OR.) The area of this rectangle can be divided into 3 equal rectangles

by lines parallel to OA. Each of these 3 rectangles has area 4 units. Hence the area of the rectangle $OAPB$ is 3×4 units, i.e. $OA.OB$ units.

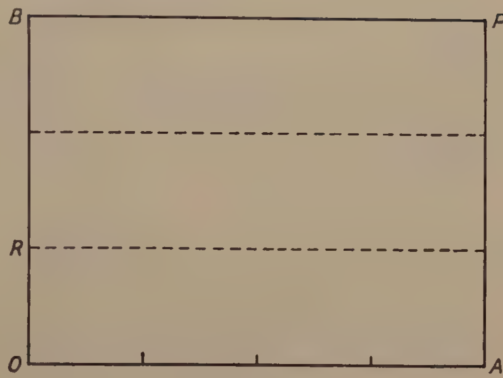


FIG. 9

Just as we write the symbol OA for the number of steps in OA, so we shall write $OAPB$ for the number of unit squares in $OAPB$, and we say that

$$OAPB = OA.OB.$$

If OA and OB are not whole numbers this statement is still true and so for any rectangle $OAPB$ we shall write $OAPB = OA.OB$. It must be remembered that this is only a short way of saying:

(The number of unit squares in) $OAPB$ is equal to (the number of steps in) $OA \times$ (the number of steps in) OB, and that the words in brackets are understood.

Example: In the rectangle $ABCD$, $AB = 3\frac{1}{4}$ and $AD = 8$. Calculate $ABCD$.

$$ABCD = AB.AD = 3\tfrac{1}{4} \times 8 = 26.$$

EXERCISE 3A

1. In the rectangle $PQRS$, $PQ = 2\frac{1}{2}$, $PS = 3\frac{1}{5}$. Calculate $PQRS$.

2. In the square $DEFG$, $DE = 5\frac{1}{2}$. Calculate $DEFG$.

3. In the rectangle $WXYZ$, $WX = 3{\cdot}14$ and $WZ = 2{\cdot}7$. Calculate $WXYZ$.

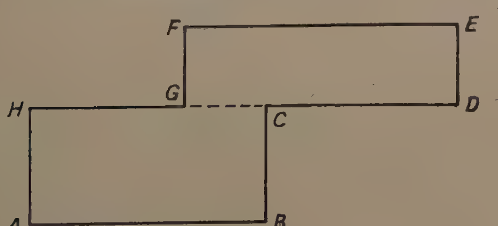


FIG. 10

4. In Fig. 10, all the angles are right-angles, H, G, C, D are in a straight line, $AB = 6$, $AH = 3$, $FG = 2$ and $FE = 7$. Calculate $ABCDEFGH$.

5. Give expressions for $OAQP$, $ABSQ$ and $OBSP$ (Fig. 11).

Use these expressions to prove that $OP.OB = OP.OA + OP.AB$.

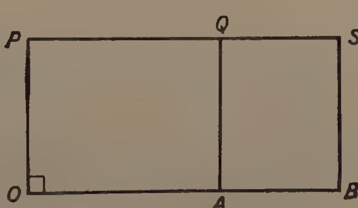


FIG. 11

Addition and Subtraction of Areas

Each of these diagrams shows two areas, ABX and ABY, with a common side, AB. When two areas share a common side in this manner we shall use the word "base" and we shall refer to AB as the common base of the two triangles.

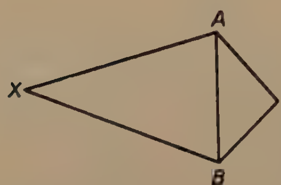


FIG. 12

FIG. 13

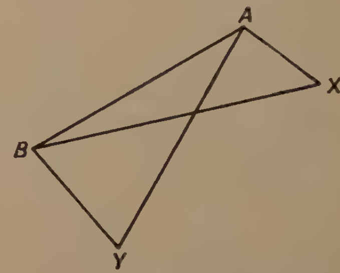


FIG. 14

In Fig. 12 the areas do not overlap and $AXBY = \triangle ABX + \triangle ABY$. In Fig. 13, ABY is within AXB and forms part of it. In this case, $AXBY = \triangle ABX - \triangle ABY$. In Fig. 14, $AXBY$ does not enclose an area, as BX crosses AY, and neither $\triangle ABX + \triangle ABY$ nor $\triangle ABX - \triangle ABY$ is equal to a single area in the figure.

It is often necessary to express an area as the sum or difference of two areas, and equations like the two above are very common. The answers to the following problems should all contain equations of this type.

EXERCISE 3B

1. In Fig. 15, express $AXBY$ as the sum of two triangles.
2. In the same figure, express $AXBY$ as the difference of two triangles.
3. In Fig. 16, express $\triangle PQR$ as the sum of two triangles.

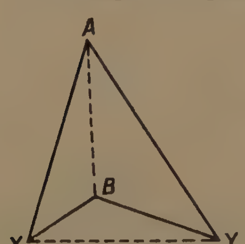


FIG. 15

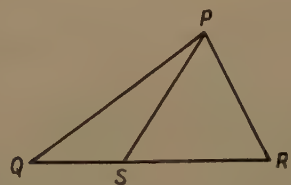


FIG. 16

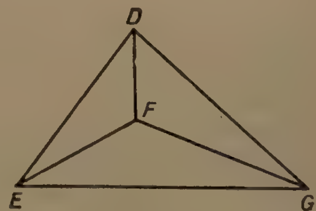


FIG. 17

4. In Fig. 17, express $\triangle DEG - \triangle FEG$ as a single area.
5. In Fig. 17, express $\triangle DEF + \triangle DFG$ as a single area.
6. In Fig. 17, prove that $\triangle DEF + \triangle EFG = \triangle DEG - \triangle DFG$.

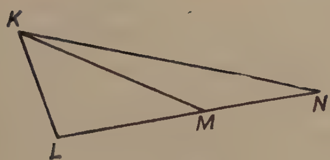


FIG. 18

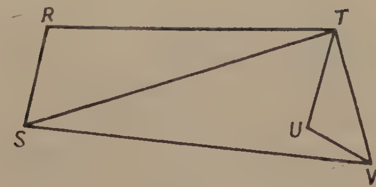


FIG. 19

7. In Fig. 18, express $\triangle KMN$ as the difference of two areas.

8. In Fig. 19, express $\triangle RST + \triangle TSV - \triangle TUV$ as a single area.

9. In Fig. 19, express $RSVT - \triangle RST - \triangle TUV$ as a single area.

The Area of a Triangle

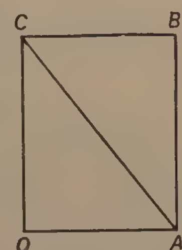


FIG. 20

The diagonal of a rectangle divides it into two triangles of equal area, i.e. each triangle is half the area of the rectangle. In Fig. 20, we write $\triangle OAC = \frac{1}{2}OABC = \frac{1}{2}OA.OC$ and we can measure the area of the right-angled triangle OAC as soon as we can find OA and OC.

In Fig. 21*a*:

$$\begin{aligned}\triangle ABC &= \triangle ABN + \triangle ACN\\ &= \tfrac{1}{2}ANBP + \tfrac{1}{2}ANCQ\\ &= \tfrac{1}{2}(ANBP + ANCQ)\\ &= \tfrac{1}{2}BCQP\end{aligned}$$

In Fig. 21*b*:

$$\begin{aligned}\triangle ABC &= \triangle ABN - \triangle ACN\\ &= \tfrac{1}{2}ANBP - \tfrac{1}{2}ANCQ\\ &= \tfrac{1}{2}(ANBP - ANCQ)\\ &= \tfrac{1}{2}BCQP\end{aligned}$$

In Fig. 21*c*:

$$\begin{aligned}\triangle ABC &= \triangle ACN - \triangle ABN\\ &= \tfrac{1}{2}ANCQ - \tfrac{1}{2}ANBP\\ &= \tfrac{1}{2}(ANCQ - ANBP)\\ &= \tfrac{1}{2}BCQP\end{aligned}$$

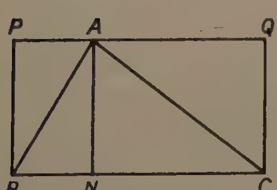


FIG. 21*a*

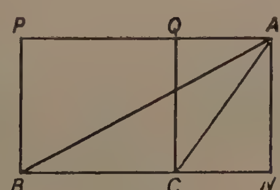


FIG. 21*b*

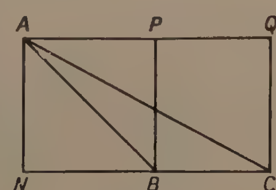


FIG. 21*c*

Each of these diagrams is constructed from the triangle ABC by following the same instructions. We draw the line through A parallel to BC and we draw BP and CQ perpendicular to this line. The rule for any triangle is: through one of the vertices draw the line parallel to the opposite side and through the ends of this side draw the perpendiculars to this side. The rectangle is formed by these two parallels and these two perpendiculars.

Each of these diagrams (Fig. 22*a*/*b*) shows a triangle XYZ and a rectangle $AXYB$ with double the area of the triangle. The side XY is called the base of the triangle to indicate that it is a side of another area in which we are interested. The line through the vertex (Z) perpendicular to the base (XY) is called the *altitude* of the triangle (for the base XY). The altitude is not drawn, but XA and YB are each equal in length to the altitude.

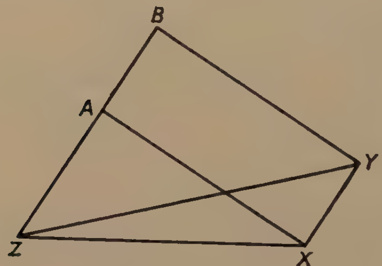


FIG. 22*a*

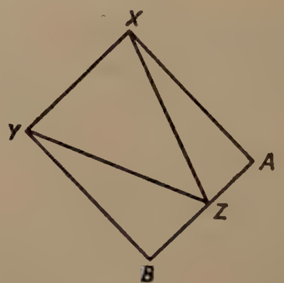


FIG. 22*b*

Thus, to find the area of a triangle, we need a base and its altitude, and the area of the triangle is half the product of these two lengths. In the diagrams we write:

$$XYZ = \tfrac{1}{2}AXYB = \tfrac{1}{2}XY.XA$$

and we note that XY is the (side selected as) base of the triangle, and XA is the altitude of the triangle (for that base). The words in brackets, as usual, are normally omitted but it is to be remembered that they are there.

Triangles With Equal Areas

Fig. 23 shows two triangles, ABP and ABQ having a common base AB. The line joining their vertices, PQ, is parallel to the base. It follows that the triangles have equal areas. This is the most important case in Geometry of two equal areas, and where there are parallel lines, there are equal areas.

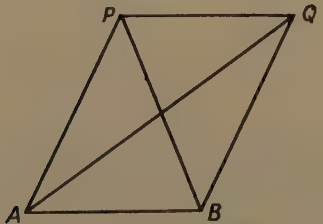


FIG. 23

The result can be given in words as follows:

If two triangles have a common base and the line joining their vertices is parallel to this base, then the triangles are equal in area.

Using letters it is even more simple:
If $AB \parallel PQ$, then $\triangle ABP = \triangle ABQ$.
Also, if needed:
$\triangle PQA = \triangle PQB$.

(Note that the two triangles contain the same four letters as the parallels; it is very bad practice to offer $PR \parallel AB$ as a reason for $\triangle ABP = \triangle ABQ$,

even though Q lies on PR. As Q is not mentioned in the reason it is not fair to deduce something about Q from this reason.)

Fig. 24 shows two triangles AOX and BOX with a common side (OX) AOB is a straight line and $AO = BO$.

Using AO as base of $\triangle AOX$, the altitude is XM, the perpendicular to AO.

Using BO as base of $\triangle BOX$, the altitude is also XM.

Hence

$\triangle AOX = \frac{1}{2}AO.MX = \frac{1}{2}BO.MX = \triangle BOX.$

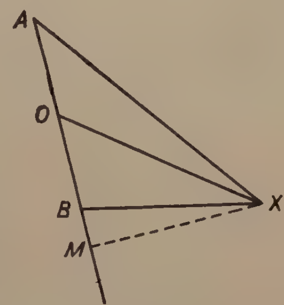


FIG. 24

The diagram, without XM, could also be regarded as a triangle XAB and O, the mid-point of AB. Such a line as OX—joining a vertex of a triangle to the mid-point of the opposite side—is called a *median* of the triangle, and as OX divides $\triangle XAB$ into two equal parts, we may say that:

The area of a triangle is bisected by a median.

(The word "bisect" is strictly reserved for dividing into two *equal* parts.)

EXERCISE 3C

1. ABC is a triangle and D is the mid-point of AB. Through A is drawn a line AX, parallel to DC. Prove that $\triangle XDC = \triangle BDC$.

2. M is the mid-point of the line PQ. PX, MN and QY are three parallel lines (to each other). Prove that $\triangle XMN = \triangle YMN$.

3. K is the mid-point of the median AM of a triangle ABC. Prove that $\triangle AKB = \frac{1}{4}\triangle ABC$. Prove also that $\triangle BKC = \frac{1}{2}\triangle ABC$.

4. $ABCD$ is a parallelogram (i.e. $AB \parallel DC$ and $AD \parallel CB$). Using these two pairs of parallels, prove that $\triangle ABC = \triangle ADC$.

5. $PQRS$ is a quadrilateral. The line through S parallel to the diagonal PR is drawn and meets QP (produced) at X. Prove that $\triangle XRQ = PQRS$ (in area).

6. $ABCD$ is a quadrilateral in which AB is parallel to DC. The diagonals AC and BD meet at X. Prove that $\triangle AXD = \triangle BXC$.

7. KLM is a triangle, X is the mid-point of KM and a line XY is drawn parallel to LM. Prove that $\triangle YLM = \triangle KLX$.

8. $ABCDE$ is a pentagon. A line is drawn through B parallel to AC and a line through E parallel to AD. These two lines meet CD (produced) at X and Y. Prove that $\triangle AXY = ABCDE$.

The Constant Area Locus

(1) ABC is a given triangle. (2) $\triangle PBC = \triangle ABC$.
What is the locus of P?

Let us examine this problem closely. The first sentence states that the locations of A, B and C are given, i.e. they are fixed points. The second sentence partially locates P, and P has many possible positions. The third sentence asks for the pattern formed by these possible positions of P.

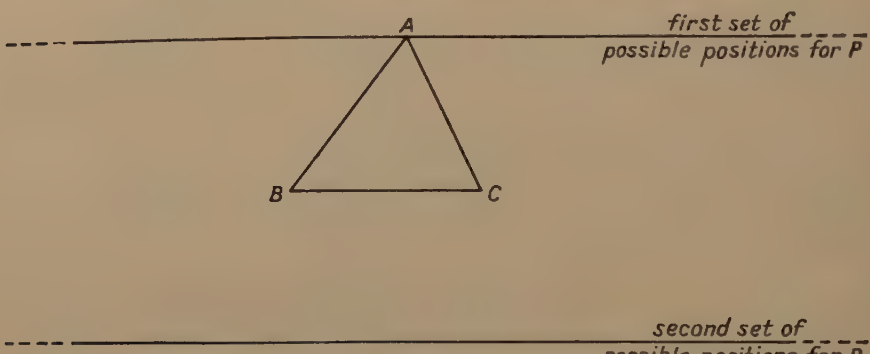


FIG. 25

There is one set of possible positions (of P) on the same side of BC as A, and there is another set on the other side of BC (Fig. 25). If P belongs to the first of these sets, then AP is parallel to BC (since $\triangle PBC = \triangle ABC$). The pattern formed by this set is thus the line through A parallel to BC. The other set also forms a line parallel to BC, at the same distance on the other side. Thus the answer to our question is (if P is partially located by the fact that $\triangle PBC = \triangle ABC$):

The locus of P is two lines parallel to BC, one of them going through A and the other at an equal distance on the other side of BC.

If, then, we are given two vertices of a triangle and also its area, *the third vertex is partially located as lying on one of* two parallel lines. The words in italics should be replaced by the words *the locus of the third vertex is*. . . . Try writing this sentence using these words.

This result may be used freely in plotting a point when one of the facts known about it is the area of the triangle which it forms with two known points.

Example: Q and R are two given points 4 inches apart. Plot P from the facts

(1) $\triangle PQR = 6$ in^2

(2) $PQ = 5$ in

The locus of P from fact (1) is two lines, each parallel to QR and 3 in from QR (Fig. 26*a*).

The locus of P from fact (2) is the circle with centre Q and radius 5 in (Fig. 26*b*).

If these two loci are drawn, then P is any one of the four points in which they intersect (Fig. 26*c*).

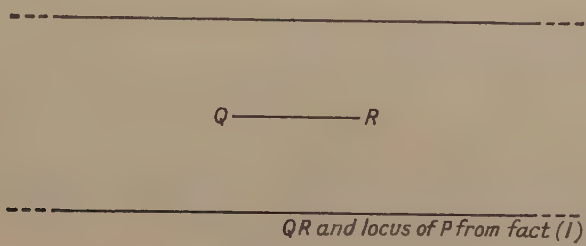


FIG. 26*a*

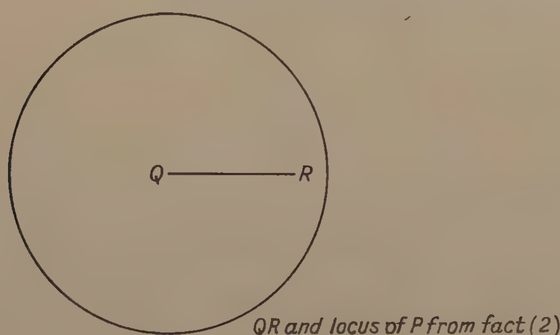


FIG. 26*b*

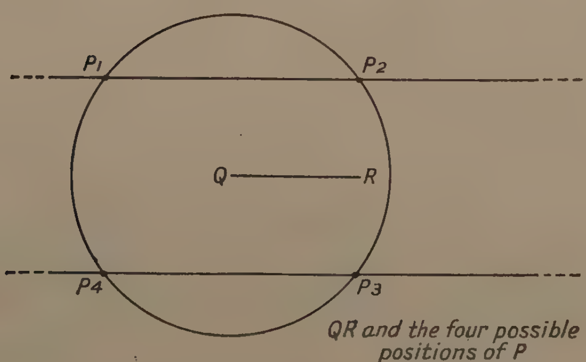


FIG. 26*c*

EXERCISE 3D

Mensuration

1. Draw a triangle ABC in which $AB = 12$ cm, $AC = 9$ cm and $\hat{A} = 52°$ Construct the perpendicular from C to AB and find the area of the triangle.

2. Calculate the length of the perpendicular from B to AC in Question 1 and check by drawing and measuring the perpendicular.

Calculation

3. $OABC$ is a square of side 8 in and P is a point inside the square such that $\triangle OCP = 20$ in². Calculate the lengths of the perpendiculars from P to OC and to AB.

4. Q is a point inside the square of Question 3, such that $\triangle OAQ = 24$ in². Calculate the area of $\triangle BCQ$.

5. R is the point inside the square of Question 3 whose (perpendicular) distances from OA and OC are 5 in and $3\frac{1}{2}$ in, respectively. Calculate the areas of $\triangle ORA$, $\triangle ORC$ and $\triangle ARC$.

6. ABC is a triangle in which $AB = AC = 7{\cdot}5$ cm and $BC = 5$ cm. AM is the perpendicular from A to BC. Calculate AM by Pythagoras' theorem and hence calculate the area of $\triangle ABC$.

7. DEF is a triangle in which $DE = DF = 13$ cm, and $EF = 10$ cm. Calculate the area of $\triangle DEF$.

Construction

8. BC is a line 4 in long. Plot a point A from the facts (1) $A\hat{B}C = 52°$ and (2) $\triangle ABC = 7$ in².

9. OAB is a triangle in which $OA = 7$ cm, $OB = 6$ cm and $A\hat{O}B$ is a right-angle. Plot a point P from the facts (1) $\triangle OAP = 14$ cm² and (2) $\triangle OBP = 15$ cm².

10. PQR is a triangle in which $PQ = 6$ cm, $PR = 10$ cm and $QR = 12$ cm. Plot a point X from the facts (1) $\triangle XPR = 40$ cm². (2) $QX = 6$ cm.

11. AB is a line 3·4 in long. Plot a point C from the facts (1) $\triangle ABC = 17$ in² and (2) $CA = CB$.

12. FGH is a triangle in which $FG = FH = 6$ cm and $GH = 8$ cm. Plot two positions for a point K from the facts (1) $\triangle KGH = \triangle FGH$ and (2) $GK = GH$.

Further Equal-Area Constructions

A polygon, we have seen, encloses an area, and it is always possible to construct a triangle which encloses an equal area. We will first show how

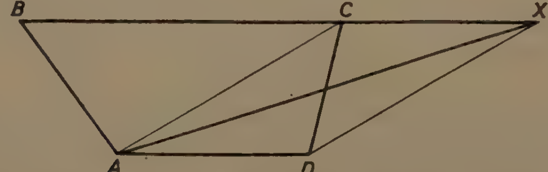


FIG. 27

to construct a triangle enclosing the same area as a given quadrilateral. $ABCD$ is the given quadrilateral (Fig. 27).

1. Draw the diagonal AC.
2. Through D draw the line parallel to AC.
3. Produce BC to meet this parallel in X.

Then $\triangle ABX =$ quadrilateral $ABCD$ (in area).

In Step 1 we may choose either diagonal.

In Step 2 the parallel to the diagonal may be drawn through either of the other two vertices of the quadrilateral.

In Step 3 we might also have produced BA to cut the parallel at Y and the equivalent triangle would then have been BCY.

We can prove $\triangle ABX =$ quad. $ABCD$ as follows:

$$\begin{aligned}\triangle ABX &= \triangle ABC + \triangle ACX \\ &= \triangle ABC + \triangle ACD \; (DX \parallel AC) \\ &= ABCD\end{aligned}$$

Further examples of this construction are given and the reader should practise this construction, freehand, on quadrilaterals of his own choosing. The construction lines are faintly drawn.

Referring to the original figure, any point Y on the line DX, taken with A, B and C would give a quadrilateral equal in area to $ABCD$. The triangle ABX is a special case of such a quadrilateral ($ABCX$) which has two of its sides (BC and CX) in the same straight line.

In a similar manner, given a 5-gon, $ABCDE$, a 4-gon of equal area can be obtained by constructing Y, the point where CD produced meets the line through E parallel to AD. The 4-gon $ABCY$ will have an area equal to $ABCDE$.

A 6-gon can similarly be reduced to a 5-gon of equal area, and so on. Thus, it is possible, though often laborious, to reduce a polygon with any number of sides to a triangle of equal area. Fig. 28 shows a 5-gon $ABCDE$ and an equivalent triangle AXY. BX is parallel to AC and EY is parallel to AD.

As an exercise, prove that $\triangle AXY = ABCDE$.

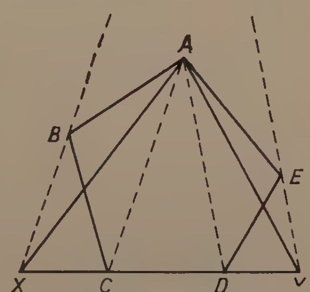


FIG. 28

Equal-Area Triangle from a Given Triangle

Given a triangle ABC and a point D in BC.

Plot a point X from the facts (1) X lies in BA produced and (2) $\triangle BDX = \triangle ABC$.

Fact (2) can be expressed in different ways.

(2a) $\triangle BDA + \triangle ADX = \triangle BDA + \triangle ADC$.

(2b) $\triangle ADX = \triangle ADC$.

(2c) $CX \parallel AD$.

Fact (2c) enables us to plot X by drawing the line through C parallel to AD, and X is the point where this parallel meets BA produced (Fig. 29).

If D were in BC produced, the same construction (draw the line through C parallel to AD) would give the required X, but in this case X would lie between A and B.

This construction enables us to replace a given triangle (ABC) by an equivalent triangle (BDX) on a given base (BD).

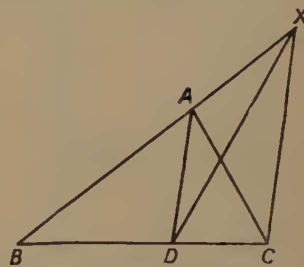


FIG. 29

Example: ABC is a given triangle; M is the mid-point of BC; P is a given point in BM. Plot the point X from the facts (1) X is in AC and (2) PX bisects $\triangle ABC$.
{Fact (2) can be expressed in different ways: (2a) $\triangle CPX = \frac{1}{2}\triangle ABC$; (2b) $\triangle CPX = \triangle AMC$; (2c) $\triangle CMX + \triangle MXP = \triangle CMX + \triangle MXA$; (2d) $\triangle MXP = \triangle MXA$; (2e) $MX \parallel PA$. Fact (2e) enables us to plot X.}

Challenge:

Given a triangle ABC and a point P in BC (near the mid-point of BC). Draw the two lines through P which will divide the triangle into three equal parts. (Start with E and F, the points which divide BC into three equal parts.)

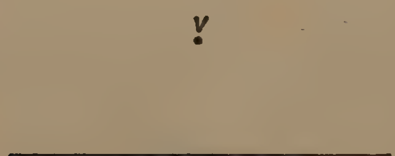


FIG. 30*a*

FIG. 30*b*

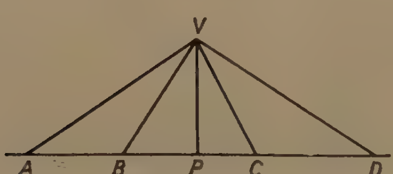


FIG. 30*c*

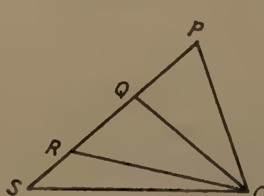


FIG. 30*d*

If we are given a straight line and a point (Fig. 30*a*) we can form a triangle by joining the point to two points on the line (Fig. 30*b*). We can repeat this with another pair of points in the line and obtain two triangles. Fig. 30*c* shows two such triangles, VAB and VCD. It also shows the perpendicular, VP, from V to the line.

Now $\triangle VAB = \frac{1}{2}AB.VP$ and $\triangle VCD = \frac{1}{2}CD.VP$.

$$\text{Hence } \frac{\triangle VAB}{\triangle VCD} = \frac{\frac{1}{2}AB.VP}{\frac{1}{2}CD.VP} = \frac{AB}{CD}$$

Similarly, in Fig. 30*d* we may write

$$\frac{\triangle OPQ}{\triangle ORS} = \frac{PQ}{RS} \text{ and } \frac{\triangle OPR}{\triangle ORS} = \frac{PR}{RS}$$

As an exercise write similar equations for

$$\frac{\triangle OPS}{\triangle ORQ}, \quad \frac{\triangle OQS}{\triangle OPQ}, \quad \frac{PR}{QS}, \quad \frac{RS}{PS}$$

Equations of this kind are useful starting points in calculations and deductions. Expressions like $\frac{\triangle OPQ}{\triangle ORS}$ and $\frac{PQ}{RS}$ are called ratios, the first being the ratio of (the areas of) two triangles, and the second the ratio of (the lengths of) two lines. The value of a ratio is a number, viz. the number of times that the lower member is contained in the upper one.

EXERCISE 3E

1. In Fig. 31, given $AB = 8$, $AC = 6$, $AX = 3$ and $AY = 4$, calculate:

(a) $\frac{\triangle AXC}{\triangle ABC}$ (*Answer:* $\frac{\triangle AXC}{\triangle ABC} = \frac{AX}{AB} = \frac{3}{8}$)

(b) $\frac{\triangle ABC}{\triangle BXC}$

(c) {using the results of (a) and (b)} $\frac{\triangle AXC}{\triangle BXC}$

(d) $\frac{\triangle BXC}{\triangle ABC}$ (e) $\frac{\triangle ABC}{\triangle ABY}$

(f) $\frac{\triangle BXC}{\triangle ABY}$ (g) $\frac{\triangle AXC}{\triangle AYB}$

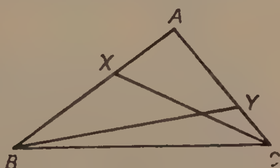


FIG. 31

2. From Fig. 32, express as the ratio of two lines:

(a) $\frac{\triangle PSR}{\triangle SQR}$ (b) $\frac{\triangle PSR}{\triangle PQR}$

(c) $\frac{\triangle PQR}{\triangle PTQ}$ (d) $\frac{\triangle PST}{\triangle SQT}$ (e) $\frac{\triangle PST}{\triangle PSR}$

3. Use the results of Question 2 to deduce that:

$$\frac{\triangle PST}{\triangle PQR} = \frac{PS.PT}{PQ.PR}$$

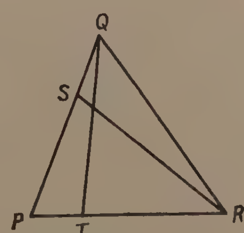


FIG. 32

4. Express as the ratio of two triangles:

(a) $\frac{PS}{SQ}$ (b) $\frac{PT}{PR}$ (c) $\frac{PS}{PQ}$

5. By expressing $\frac{\triangle PQT}{\triangle PQR}$ and $\frac{\triangle PQR}{\triangle PSR}$ as ratios of lines deduce that

$$\frac{\triangle PQT}{\triangle PSR} = \frac{PQ.PT}{PS.PR}$$

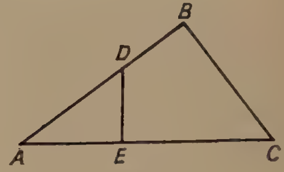


FIG. 33

6. Deduce in Fig. 33 that:

$$\frac{\triangle ABC}{\triangle ADE} = \frac{AB.AC}{AD.AE}$$

7. Given $AD = 5$, $DB = 3$, $AE = 4$ and $EC = 6$, calculate $\dfrac{\triangle ABC}{\triangle ADE}$

8. By expressing $\dfrac{\triangle UYZ}{\triangle XYZ}$ and $\dfrac{\triangle VYZ}{\triangle XYZ}$ (Fig. 34) as line ratios, deduce that if $\dfrac{YU}{XY} = \dfrac{ZV}{XZ}$ then UV is parallel to YZ.

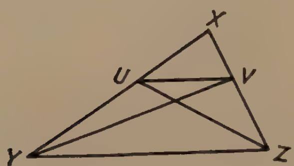


FIG. 34

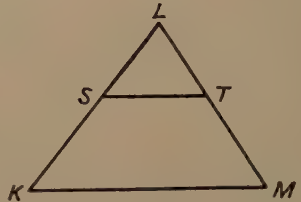


FIG. 35

9. Given that ST is parallel to KM (Fig. 35), deduce that $\dfrac{\triangle LST}{\triangle LKM} = \dfrac{LS^2}{LK^2}$

(Express $\dfrac{\triangle LST}{\triangle LKT}$ and $\dfrac{\triangle LKT}{\triangle LKM}$ as line ratios, and use the fact that $\dfrac{LS}{LK} = \dfrac{LT}{LM}$ since ST is given parallel to KM.)

In Question 9 the two triangles forming the first ratio are similar, and the result may be stated in the words:

The ratio of the areas of two similar triangles is equal to the ratio of the squares of corresponding sides.

This fact is true of any two similar triangles and it may freely be used in other calculations and deductions.

CHAPTER 4

POLYGONS AND QUADRILATERALS

IF A NUMBER of points are plotted, there are many ways in which a journey can be made, starting at one of the points, and after visiting each of the other points in turn, returning to the starting-point. If the journeys between points are straight lines, the path followed may be called a POLYGON. The points are called the *vertices*, and the straight lines are called the *sides* of the polygon. We shall reserve the name polygon for a path which does not cross itself at any point. No matter how many points are taken, it is always possible to visit all the points in turn, returning to the starting point, by a path which does not cross itself at any point. (This statement can be proved.)

Every polygon will have as many sides as it has vertices. Each vertex is the intersection of two sides and two such sides form two angles. It is easy to see that one of these angles is "inside" the polygon and the other "outside". The inside angles will be called the *interior* angles. (Don't be in too big a hurry and think that we shall call the others the *exterior* angles; we shall not call them anything, but leave them entirely out of our discussion.)

Polygons are distinguished by the number of vertices (or sides) which they possess. Thus, a polygon with 7 vertices is called a 7-gon and one with 18 vertices is called an 18-gon. A triangle could be called a 3-gon although we shall usually stick to the name triangle for such a figure. It is also interesting to note that we can draw a 2-gon. (Plot 2 points; start from one, visit the other and return to your starting-point. You have now described a 2-gon. Its interior angles are both zero.)

We are now able to establish two remarkable facts. First, that the sum of the interior angles of any polygon is equal to the sum of the interior angles of any other polygon having the same number of vertices; and, secondly, we can prophesy what this sum will be for a given number of vertices. For, given a polygon with any number of vertices, we can always find a vertex such that the *next but one* can be reached by a straight line lying inside the polygon. (This statement is not true if the path crosses itself, but this case is excluded in our description of a polygon.)

If we cut along such a line and remove the triangle which is cut off, we are left with a polygon with 1 fewer sides than before. Also, the sum of the interior angles is 2 right-angles less than before. Thus the sum of the interior angles of a polygon is 2 right-angles more than the sum of the interior angles of a polygon with 1 fewer side; and this argument can be repeated, decreasing the number of sides by 1 at each step and the sum of the angles by 2 right-angles.

Now since the sum of the angles of a 3-gon is 2 right-angles
the sum of the angles of a 4-gon is $2 + 2$ or 4 right-angles
and the sum of the angles of a 5-gon is $4 + 2$ or 6 right-angles
and the sum of the angles of a 6-gon is $6 + 2$ or 8 right-angles

We notice that the sum, in right-angles, is always 4 less than twice the number of sides, so we conclude that:

The sum of the angles of an N-gon is $(2N - 4)$ right-angles.

This statement may be freely used about any polygon, except when it is required to prove the statement. E.g. if we are interested in the angles of an 8-gon, we may start:

The sum of the angles of an 8-gon is $(2 \times 8 - 4)$ or 12 right-angles.

When a polygon has all its sides of the same length *and* all its angles of the same size, the polygon is said to be a *regular* polygon. It is an easy calculation to find the size of each angle of any regular polygon when we know the number of its sides; e.g. for a regular 24-gon, we say (freely, as above):

The sum of the angles of a 24-gon is $(2 \times 24 - 4)$ or 44 right-angles, or $44 \times 90°$. Hence each angle $= \dfrac{44 \times 90}{24}$ degrees, or 165°.

EXERCISE 4A

1. Seven of the angles of an 8-gon are each 133°. Calculate the eighth angle.

2. Six of the angles of a 9-gon are each 139°. Calculate the size of the other three, given that they are of equal sizes.

3. Calculate the interior angle of a regular 20-gon.

4. K, L and M are consecutive vertices of a regular 12-gon. Calculate the angles KLM and KML.

5. The interior angle of a regular n-gon is 156°. Find the numerical value of n.

6. Using your protractor, draw a regular 10-gon with sides $1\frac{1}{2}$ inches in length.

Quadrilaterals

A 4-gon is a figure bounded by four straight lines. It is usually called a *quadrilateral*. A quadrilateral has four vertices and the lines joining opposite pairs of vertices are called the *diagonals* of the quadrilateral. A quadrilateral is usually described by naming *in order* the four vertices through which we pass in going round the quadrilateral. This quadrilateral (Fig. 1*a*) could be named *ABCD* or *DCBA* but

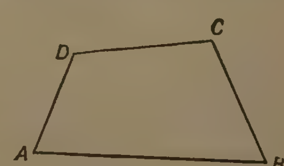


FIG. 1*a*

not ABDC. The name of the other quadrilateral Fig. 1*b*) is *PQSR* and *not PQRS*.

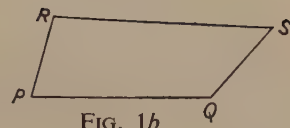


FIG. 1*b*

Given a quadrilateral ABCD (Fig. 2*a*), *to construct another quadrilateral* (*WXYZ*) *having the same shape.*

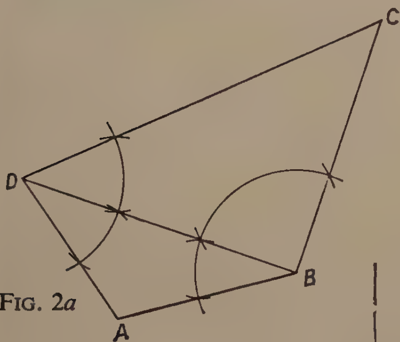


FIG. 2*a*

1. Plot two of the vertices. (We have chosen X and Z and they may have any positions we please.)

2. Copy the angle DBC at X (i.e. draw the line through X which makes with XZ an angle equal to $D\hat{B}C$. Fig. 2*b* shows this done with ruler and compasses).

3. Copy the angle BDC at Z. (The two lines drawn in steps 2 and 3 locate Y, as seen in Fig. 2*c*.)

4. Copy the angle ADB at Z (Fig. 2*d*).

5. Copy the angle DBA at X. (The two lines in steps 4 and 5 locate W, as seen in Fig. 2*e*.)

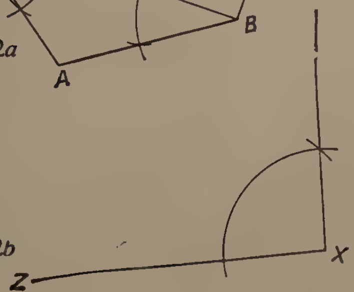


FIG. 2*b*

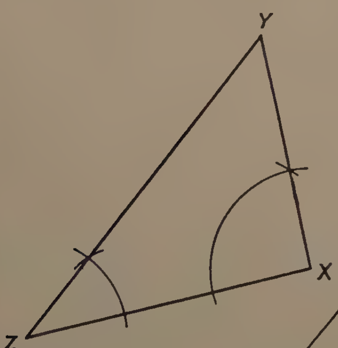


FIG. 2*c*

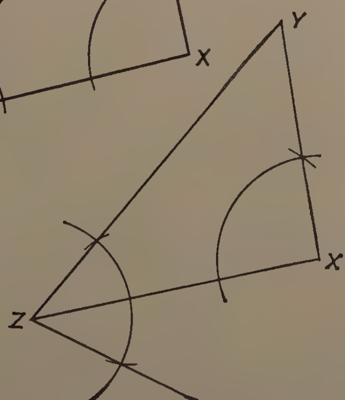


FIG. 2*d*

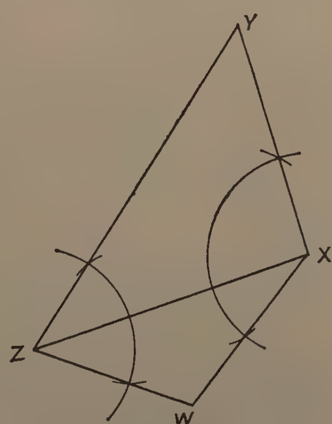


FIG. 2*e*

If it is required to make $WXYZ$ the same size as well as the same shape as $ABCD$, step 1 would become: Plot X and Z so that $XZ = BD$ and the rest would follow as before.

Instead of being given a picture of the quadrilateral $ABCD$ it is usual to give the measurements of the side and angles to be copied, and we see that five such measurements are required in order to draw the quadrilateral. The number five is important here; with fewer than five measurements given it would not be possible to locate all four vertices completely; with exactly five we are able to plot the four vertices; a sixth measurement chosen at random would have little chance of fitting the four points already plotted from the first five measurements.

The five measurements needed to determine (or fix, or specify) a quadrilateral may be distances (between two of the vertices), or angles (between lines joining the vertices), but there must be no wasted measurements, e.g. if the angles BAC and CAD were given it would be a wasted measurement to give BAD also, for $B\hat{A}C + C\hat{A}D = B\hat{A}D$.

In the following examples the first measurement given is used to plot two of the vertices. Each of the other two vertices is then plotted by using *two* of the other given facts. It will be remembered that when two points have been plotted, each additional point requires two facts to locate it.

EXERCISE 4B

Draw quadrilaterals from the given measurements (or facts). The facts are given in the order in which they should be used.

1. (Quad. $ABCD$) $AB = 9$ cm, $B\hat{A}D = 100°$, $AD = 6$ cm, $A\hat{B}C = 120°$, $BC = 7{\cdot}2$ cm.

2. (Quad. $ABCD$) $AB = 4$ in, $AD = 3$ in, $BD = 4$ in, $A\hat{D}C = 127°$, $A\hat{B}C = 90°$.

3. (Quad. $CDEF$) $CD = 8$ cm, $D\hat{C}F = 120°$, $C\hat{D}F = 20°$, $C\hat{D}E = 110°$, $FE \parallel CD$.

4. (Quad. $CDEF$) $CD = 4$ in, $CF = 2$ in, $DF = 3$ in, $DE = 4$ in, $CE = 5$ in.

5. (Quad. $EFGH$) $FG = 10$ cm, $F\hat{G}H = 90°$, $GH = 10$ cm, $FE \parallel GH$, $HE \parallel FG$.

6. (Quad. $WXYZ$) $WX = 2$ in $= XY$, $WY = 3$ in, $WZ = 2$ in $= YZ$.

7. (Quad. $ABCD$) $AB = 3$ in, $B\hat{A}C = 30°$, $A\hat{B}C = 60°$, $A\hat{C}D = 30°$, $AD = 1{\cdot}3$ in. (What would happen if $AD = 1$ in?)

8. *A challenge.* (Quad. $KLMN$) (1) $LM = 9$ cm, (2) $MN = 9$ cm, (3) $NL = 9$ cm, (4) $KL = KN$, (5) $MK = 12$ cm.

(When L, M, N have been plotted, what is the locus of K from fact 4, and what is the locus of K from fact 5?)

Quadrilaterals Having Special Properties

Any fact which helps to determine a quadrilateral (or any other figure) may be called a property of the quadrilateral (or figure), but we shall reserve the word "property" to mean a fact which is not actually a measurement.

Thus, $AB = 3$ in is not considered to be a property of the quadrilateral $ABCD$, but if $AB \parallel DC$, then the quad. $ABCD$ has the property that two of its sides are parallel; if $AB = AD$, then $ABCD$ has the property that two of its adjacent sides are equal; and if $B\hat{A}D = B\hat{C}D$, then $ABCD$ has the property that two opposites angles are equal.

Some properties relate to the diagonals of the quadrilateral. Thus we can have a quadrilateral with the property that its diagonals are equal, or with the property that one of the two diagonals bisects the other. (Bisect means divide into two *equal* parts.)

A quadrilateral may have more than one property, e.g. it may have two parallel sides and equal diagonals (two properties); or it may have two sides parallel and the other two equal and perpendicular, i.e. to each other (three properties). Fig. 3 shows such a quadrilateral $ABCD$ in which two sides are parallel (AB and DC) and the other two (AD and BC) are perpendicular to each other.

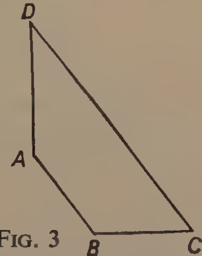


FIG. 3

We shall find that when a quadrilateral is given certain properties, then it cannot be prevented from having certain others, i.e. it is impossible to draw a quadrilateral, having the given properties, and not having the others. We shall find that it is impossible to draw a quadrilateral having the properties

(i) having two adjacent sides equal,

(ii) having the other two sides equal,

which *does not also* have the property that its diagonals are perpendicular.

Of these properties, the former are referred to as the "given" properties and the latter as the "deduced" properties. We shall be concerned for a time in finding what properties of quadrilaterals can be deduced from certain given properties.

Names are given to quadrilaterals according to the special properties which they possess. It is important to learn these names and the properties to which they refer.

Properties of a Trapezium. A quadrilateral having two of its sides parallel is called a TRAPEZIUM. Thus the "given" property of a trapezium is that two of its sides are parallel. A trapezium has a property, concerned with its diagonals, which we now deduce:

In Fig. 4:
The given property is $AB \parallel DC$.
The diagonals intersect at O.
In the triangles AOB and COD,

$O\hat{A}B = O\hat{C}D$ (alt. $AB \parallel DC$)
and $O\hat{B}A = O\hat{D}C$ (alt. $AB \parallel DC$)
$\therefore \triangle AOB \,|||\, \triangle COD$

$$\therefore \frac{OA}{OB} = \frac{OC}{OD}$$

$$\therefore \frac{OA}{OC} = \frac{OB}{OD}$$

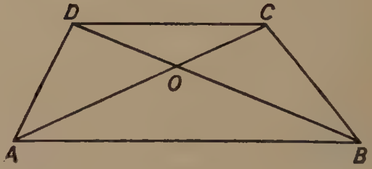


FIG. 4

Now $\frac{OA}{OC}$ is the ratio in which BD divides AC, and $\frac{OB}{OD}$ is the ratio in which AC divides BD; so we have as the "deduced" property:

Each diagonal divides the other in the same ratio.

From the similar triangles we can also deduce $\frac{OA}{AB} = \frac{OC}{CD} \therefore \frac{OA}{OC} = \frac{AB}{CD}$ so that the complete deduced fact is, in symbols,

$$\frac{OA}{OC} = \frac{AB}{CD} = \frac{OB}{OD} \text{ or, in words:}$$

Each diagonal divides the other in the ratio of the parallel sides.

Trapezium with Parallel Sides Equal. No special name is given to a trapezium with this extra property, but it has a very interesting deduced property.

In any trapezium $ABCD$ with $AB \parallel DC$, we have deduced that

$$\frac{OA}{OC} = \frac{OB}{OD} = \frac{AB}{CD}$$

If the trapezium has the extra property that $AB = CD$, then each of these ratios is 1, and the deduced property of a trapezium with parallel sides equal is that:

The diagonals bisect each other, or, in other words,

The intersection of the diagonals is the mid-point of each.

Properties of a Parallelogram. A quadrilateral having two pairs of parallel sides is called a *parallelogram*. Such a quadrilateral has *two* given properties, one more than a trapezium. If the instructions for drawing a quadrilateral $ABCD$ contain the facts (1) $AB \parallel DC$ and (2) $AD \parallel BC$, then $ABCD$ is a parallelogram. The other three facts required to specify the quadrilateral may be any we wish.

The instruction to draw a quadrilateral $ABCD$ from the facts:
(1) $AB = 4$ in, (2) $B\hat{A}D = 60°$, (3) $AD = 2$ in, (4) $DC \parallel AB$ and
(5) $BC \parallel AD$, may be given as: Draw a parallelogram $ABCD$ from the facts

(1) $AB = 4$ in, (2) $B\hat{A}D = 60°$, (3) $AD = 2$ in.

There is no need to draw this figure accurately. It should be drawn freehand and the measurements guessed. The shape of the figure is the important thing. A parallelogram is also formed when we draw a pair of parallel lines and another pair crossing them.

A parallelogram, then, has two given properties and it has a number of deduced properties. We shall deduce first that its opposite sides are equal (as well as parallel).

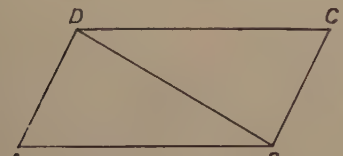


FIG. 5

$ABCD$ is a parallelogram with the diagonal BD drawn (Fig. 5).

In the triangles ABD, CBD,

$A\hat{B}D = C\hat{D}B$ (alt., $AB \parallel CD$)

$A\hat{D}B = D\hat{B}C$ (alt., $AD \parallel BC$)

$\therefore ABD \,|||\, CDB \therefore \dfrac{AB}{BD} = \dfrac{CD}{BD} \therefore AB = CD.$

Also, $\dfrac{AD}{BD} = \dfrac{CB}{BD} \therefore AD = CB$

Thus, if AB has the property of being a parallelogram, it also has the property of having both pairs of opposite sides equal. This deduced fact may be freely used about any quadrilateral which is known to be a parallelogram.

EXERCISE 4C

In the following figures, parallels are indicated by arrows and two lines bearing the same number of arrow-heads are given parallel, e.g. $KR \parallel NM$.

1. Name as many parallelograms as you can in each figure.

2. Give the names of *pairs* of equal lines using the fact that opposite sides of a parallelogram are equal.

(Write your answers as $HD = GE$, etc.)

3. In Fig. 6*b* deduce that $FG = GH$.

4. In Fig. 6*c* deduce that (a) $KR = NM$; (b) $PN = RS$; (c) $PK = MS$.

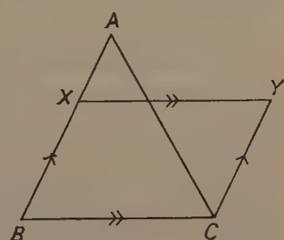


FIG. 6*a*

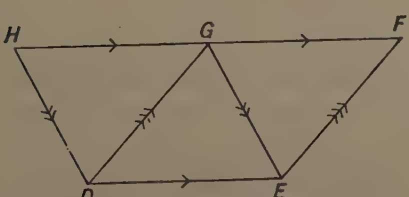


FIG. 6*b*

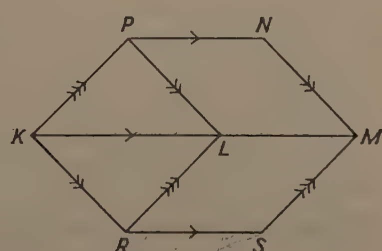


FIG. 6*c*

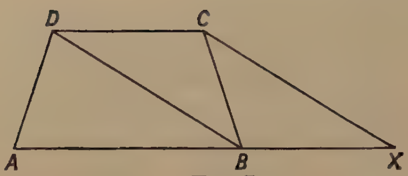


FIG. 7

Constructing a Trapezium. Fig. 7 shows a trapezium $ABCD$ and a fifth point, X, located from the facts (1) X lies in AB, (2) $CX \parallel DB$. $BXCD$ is a parallelogram (since $CX \parallel DB$ and $DC \parallel BX$),

$\therefore$ $BX = CD$ and $XC = BD$.

Now consider this problem:

Construct a trapezium $ABCD$ from the facts (1) $AB = 4$ in; (2) $CD = 2$ in; (3) $AC = 3{\cdot}5$ in; (4) $BD = 5$ in; (5) $AB \parallel DC$.

We begin by drawing AB. We can draw one locus for C from fact (3) and we can draw one locus for D from fact (4), but we cannot find a second locus for either C or D and so we are unable to locate either of them.

Fig. 7 provides the clue. We plot next the point X, located by the facts (1) X lies in AB, (2) CX ($= BD$) $= 5$ in.

C can now be located from the facts

$$(1)\ AC = 3{\cdot}5 \text{ in; } (2)\ XC\ (= BD) = 5 \text{ in.}$$

D can now be located from the facts (1) $CD \parallel BA$; (2) $BD = 5$ in (or $BD \parallel XC$ would do equally well). Thus, the trapezium is complete.

Try this construction, and note that it is impossible to construct the trapezium without first plotting X.

Quadrilaterals with Both Pairs of Opposite Sides Equal. Construct a quadrilateral $ABCD$ from the facts (1) $AB = 9$ cm; (2) $AC = 12$ cm; (3) $BC = 6$ cm; (4) $AD = 6$ cm; (5) $CD = 9$ cm.

(Plot A and B; plot C from facts (2) and (3); plot D from facts (4) and (5).)

This quadrilateral has the properties $AB = CD$ and $BC = AD$, so that it is a quadrilateral with two pairs of opposite sides equal.

We can deduce that the opposite sides are also parallel. In the quadrilateral $ABCD$, the given facts are

$$(1)\ AB = CD \text{ and } (2)\ AD = BC.$$

In the triangles ABD, BDC,

$$\frac{AB}{BD} = \frac{CD}{BD} \text{ (since } AB = CD\text{), and } \frac{AD}{BD} = \frac{BC}{BD} \text{ (since } AD = BC\text{)}$$

$\therefore \triangle ABD \parallel\!\mid \triangle BDC \therefore A\hat{D}B = D\hat{B}C \therefore AD \parallel BC$.

Also $A\hat{B}D = B\hat{D}C \therefore AB \parallel CD$.

Thus, if a quadrilateral has the given properties that both pairs of opposite sides are equal, then it has also the deduced properties that its opposite sides are parallel.

This result provides another construction for parallel lines with ruler and compasses, which requires fewer arcs than that given earlier.

To Draw a Line Through a Given Point Parallel to a Given Line. Let A be the given point and BC the given line. Plot the point D from the facts (1) $AD = BC$ and (2) $CD = AB$. Then AD will be the required line (Fig. 8). (The locus of D from fact (1) is the circle with centre A and radius BC; the locus of D from fact (2) is the circle with centre C and radius AB. D is the intersection of these two loci.)

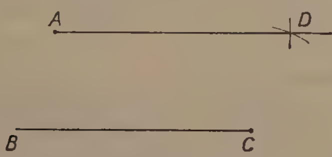


FIG. 8

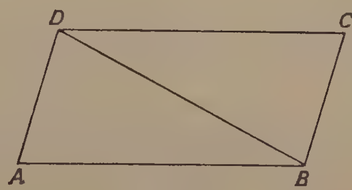


FIG. 9

Quadrilaterals with One Pair of Sides Parallel and Equal. Such a quadrilateral has two given properties, one more than a trapezium.

$ABCD$ is a quadrilateral in which the given facts are (1) $AB \parallel DC$ and (2) $AB = DC$ (Fig. 9).

In the triangles ABD and BDC,

$$A\hat{B}D = B\hat{D}C \text{ (alt., } AB \parallel DC\text{) and } \frac{AB}{BD} = \frac{DC}{BD} \text{ (since } AB = CD\text{)}$$

$$\therefore \triangle ABD \parallel\!| \triangle BDC \therefore A\hat{D}B = C\hat{B}D \therefore AD \parallel BC.$$

$\therefore$ $ABCD$ is a parallelogram (since it has two pairs of sides parallel).

We have been discussing quadrilaterals with two special properties. In the first case the two properties were two pairs of parallel sides. Such a quadrilateral we have *agreed* to call a parallelogram, and the *definition* of a parallelogram is *a quadrilateral with its opposite sides parallel.*

In the second case the two properties were two pairs of equal sides. We did not give a special name to such quadrilaterals, for we were able to deduce that such a quadrilateral is also a parallelogram.

In the third case the two properties were a pair of equal and parallel sides. Again, we did not give these a special name, for we were able to deduce that such a quadrilateral is also a parallelogram.

The word parallelogram, then, may be used for any quadrilateral which has one of these three pairs of properties, in the first case by definition, and in the other two cases, by deduction.

In each of the three preceding cases we have been given two properties of a quadrilateral and we have deduced other properties from the given ones. These results can be stated in words as follows.

1. The opposite sides of a parallelogram are equal.

(The given properties are two pairs of parallel sides; the deduced properties are two pairs of equal sides.)

2. A quadrilateral with both pairs of opposite sides equal is a parallelogram.

(The given properties are two pairs of equal sides; the deduced properties are two pairs of parallel sides.)

3. A quadrilateral with one pair of sides equal and parallel is a parallelogram.

(The given properties are a pair of equal and parallel sides; the deduced properties are two pairs of parallel sides.)

What is a Theorem?

These three statements are examples of what are called THEOREMS in Geometry. A theorem is a statement that if a figure has certain properties, then certain others can be deduced. Pythagoras' theorem, for instance, which we met in Chapter 2, is a statement that if a figure (a triangle) has a certain property (that one of its angles is a right-angle), then another property (that the sum of the squares on two of its sides is equal to the square on the third side) can be deduced.

A theorem is *proved* by deducing the required properties from the given ones. When it has been proved, it may be used to deduce properties from figures which have the given properties, and also to prove other theorems.

The proof of a theorem usually contains three paragraphs.

1. *Given.* This paragraph contains the given properties of the figure, and also the names given to the points in the figure.

2. *To Prove.* This paragraph states the properties of the figure which we are required to deduce. These properties should be given, using the names of points in the figure.

3. *Proof.* This paragraph shows as clearly as possible how the deduced properties follow from the given properties.

To illustrate, we give a proof of theorem No. 2 above.

Given: A quadrilateral $ABCD$ in which $AB = CD$ and $AD = CB$ (Fig. 10).
To prove: $ABCD$ is a parallelogram.
Proof: In the triangles ABD, BDC,

$$\frac{AB}{BD} = \frac{CD}{BD} \text{ (since } AB = CD\text{, given)},$$

$$\frac{AD}{BD} = \frac{CB}{BD} \text{ (since } AD = BC\text{, given)}.$$

$\therefore \triangle ABD$ is similar to $\triangle BDC$.

$\therefore A\hat{D}B = D\hat{B}C \therefore AD \parallel BC$

Also $A\hat{B}D = B\hat{D}C \therefore AB \parallel CD$

$\therefore ABCD$ is a parallelogram.

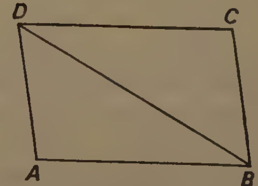


FIG. 10

A note on the sign $\therefore$. This sign is usually read "therefore" and it acts as a conjunction between two properties, or statements. Thus, when we read above $A\hat{D}B = D\hat{B}C \therefore AD \parallel BC$, the sign $\therefore$ means that a figure with the property $A\hat{D}B = D\hat{B}C$ also has the property $AD \parallel BC$. The sign should be used very carefully, and it is often wrongly used. It should be used only between two properties, of which the second is deduced from the first.

In Geometry we also have another sign ($\because$) which is used in a similar manner, but this sign connects two statements of which the *first* is deduced from the *second.* Thus we may write either $AB \parallel DC$, $AD \parallel BC \therefore ABCD$ is a parallelogram; or $ABCD$ is a parallelogram $\because AB \parallel DC$ and $AD \parallel BC$. The sign $\because$ is read as the word "because." It could also have been used in the proof given above in place of the word "since."

We use a theorem in a piece of work by going directly from its given property to its deduced property, using the $\therefore$ sign. It is a general rule that a theorem may be used freely at any time *except when we are asked to prove the theorem.*

Example: Given that $ABPQ$ and $ABXY$ are parallelograms, deduce that $PQ = XY$.

(In this example, all the points are named in the question, so there is no need for the paragraph "*Given.*")

Proof: $ABPQ$ is a parallelogram $\therefore PQ = AB$.
$ABXY$ is a parallelogram $\therefore XY = AB$.
$\therefore PQ = XY$.

But, given that $ABCD$ is a parallelogram, deduce that $AB = DC$.

In this case we are asked to prove the theorem itself, so we may *not* say "$ABCD$ is a parallelogram $\therefore AB = DC$." We should have to give the full proof of the theorem.

Theorems about the Diagonals of Parallelograms

The diagonals of a parallelogram bisect each other.

Given: A parallelogram $ABCD$ and O the intersection of the diagonals (Fig. 11).

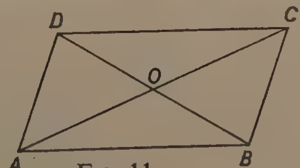


FIG. 11

To prove: $OA = OC$ and $OB = OD$.

Proof: In the triangles OAB and OCD,
$O\hat{A}B = O\hat{C}D$ ($\because AB \parallel DC$)
$O\hat{B}A = O\hat{D}C$ ($\because AB \parallel DC$)
$\therefore \triangle OAB \,|||\, \triangle OCD$.
$\therefore \dfrac{OA}{AB} = \dfrac{OC}{CD}$, but $AB = CD$ ($\because ABCD$ is a parallelogram).
$\therefore OA = OC$.
Similarly $OB = OD \therefore OA = OC$, and $OB = OD$.

The word similarly here means that we could deduce $OB = OD$ by a similar argument: $\frac{OB}{AB} = \frac{OD}{CD}$ etc. The word is used when the deduction is of the same sort as the one just completed.

A quadrilateral whose diagonals bisect each other is a parallelogram.

Given: A quadrilateral $ABCD$ whose diagonals intersect at O and such that $OA = OC$ and $OB = OD$.

To prove: $ABCD$ is a parallelogram.

Proof: In the triangles OAB and OCD,

$A\hat{O}B = C\hat{O}D$ (vertically opposite).

$\frac{OA}{OB} = \frac{OC}{OD}$ ($OA = OC$ and $OB = OD$, given).

$\therefore \triangle OAB ||| \triangle OCD \therefore O\hat{A}B = O\hat{C}D \therefore AB \parallel CD$.

Similarly* $AD \parallel BC \therefore$ *ABCD is a parallelogram.*

A quadrilateral with two adjacent sides equal and the other pair of sides also equal is sometimes called a *kite*.

Construct a quadrilateral $ABCD$ from the facts (1) $BD = 3\frac{1}{2}$ in., (2) $BA = 2$ in., (3) $DA = 2$ in., (4) $BC = 4$ in., (5) $DC = 4$ in.

Note that this quadrilateral has two adjacent sides, BA and DA, equal; it also has the other two sides, BC and DC, equal, so that this quadrilateral is a kite. We can deduce two properties of the diagonals of a kite.

$ABCD$ is a kite in which $AB = AD$ and $CB = CD$ (Fig. 12a).

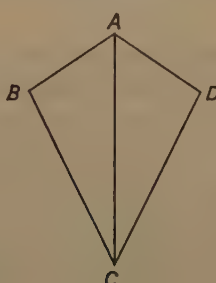


FIG. 12a

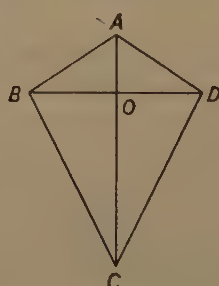


FIG. 12b

In the triangles ABC and ADC,

$$\frac{AB}{AC} = \frac{AD}{AC} \text{ (since } AB = AD\text{, given)}$$

$$\frac{BC}{AC} = \frac{DC}{AC} \text{ (since } BC = DC\text{, given)}$$

$\therefore ABC ||| ADC \therefore B\hat{A}C = D\hat{A}C$ and $B\hat{C}A = D\hat{C}A$

or, in words, *the diagonal AC bisects the angles A and C.*

* i.e. using $\triangle OAD$ and $\triangle OBC$.

If the diagonals intersect at O, we deduce a further property of the diagonals (Fig. 12*b*).

In the triangles BAO and DAO,

$B\hat{A}O = D\hat{A}O$ (proved above)

$\dfrac{AB}{AO} = \dfrac{AD}{AO}$ (since $AB = AD$)

$\therefore BAO \parallel\!| DAO \therefore B\hat{O}A = D\hat{O}A \therefore B\hat{O}A$ is a right-angle, or AC and BD are perpendicular. We have thus deduced that

The diagonals of a kite are perpendicular.

The word "kite" is not generally accepted in the same way as "parallelogram," and its deduced properties should not be used without proving them first. The word has been used here for a quadrilateral with two special properties, and it is a useful word to have in mind when dealing with such a quadrilateral.

Parallelograms Having Special Properties

A parallelogram having one of its angles a right-angle is called a RECTANGLE. A rectangle is thus a quadrilateral with *three* special properties: (1) one pair of sides parallel, (2) the other pair also parallel and (3) one of its angles is a right-angle. We can immediately *deduce* that all four angles are right-angles. We could also have a quadrilateral with the given properties of having three right-angles, and we could *deduce* that this figure would be a rectangle.

It is important to distinguish between given properties and deduced properties in proving theorems about rectangles. It may seem strange at first to be asked to *prove* that all the angles of a rectangle are right-angles, but this is quite a reasonable request.

The given properties are those numbered 1, 2 and 3 above, and the properties to be deduced are that the other three angles are right-angles. The proof of the theorem would be:

Given. A parallelogram $ABCD$, in which A is a right-angle.

To prove. Angles B, C and D are right-angles.

Proof. $AB \parallel DC$ (given) $\therefore \hat{A} + \hat{D} = 2$ rt.-angles.

$\hat{A} = 1$ rt.-angle (given) $\therefore \hat{D} = 1$ rt.-angle.

$AD \parallel BC$ (given) $\therefore \hat{A} + \hat{B} = 2$ rt.-angles.

$\hat{A} = 1$ rt.-angle (given) $\therefore \hat{B} = 1$ rt.-angle.

$\hat{A} + \hat{B} + \hat{C} + \hat{D} = 4$ rt.-angles $\therefore \hat{C} = 1$ rt.-angle.

$\therefore$ Angles B, C and D are right-angles.

A parallelogram with two adjacent sides equal is called a RHOMBUS. A rhombus is thus a quadrilateral with *three* special properties. We can immediately deduce that all four sides have the same length.

A SQUARE is a parallelogram with two additional properties, (1) having one angle a right-angle and (2) having two adjacent sides equal. Thus a square is a quadrilateral with *four* special properties.

Rectangles and rhombi (Latin plural of rhombus) are parallelograms with an additional property. Their diagonals have additional properties besides that of bisecting each other.

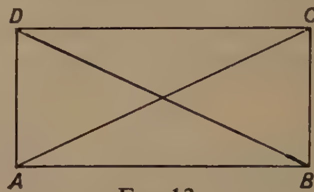


FIG. 13

The diagonals of a rectangle are equal.

Given. A rectangle $ABCD$ (Fig. 13).

To prove. $AC = BD$.

Proof. In the triangles DAB and CBA,

$D\hat{A}B = C\hat{B}A$

(rt. $\angle$s $\because$ $ABCD$ is a rectangle).

$\dfrac{DA}{AB} = \dfrac{CB}{BA}$

($DA = CB \because ABCD$ is a parallelogram).

$$\therefore \triangle DAB \,|||\, \triangle CBA \therefore \frac{DB}{AB} = \frac{CA}{BA} \therefore AC = BD$$

The diagonals of a rhombus are perpendicular.

Given. A rhombus $ABCD$ with diagonals intersecting at O (Fig. 14).

To prove. $A\hat{O}B$ is a right-angle.

Proof. $AB = AD$ (given). $OB = OD$ (AC, BD bisect each other, since $ABCD$ is a parallelogram).

In the triangles AOB, AOD,

$\dfrac{OB}{OA} = \dfrac{OD}{OA}$ ($\because OB = OD$)

$\dfrac{AB}{OA} = \dfrac{AD}{OA}$ ($\because AB = AD$)

$\therefore \triangle AOB \,|||\, \triangle AOD$

$\therefore A\hat{O}B = A\hat{O}D$

$\therefore$ *$A\hat{O}B$ is a right-angle.*

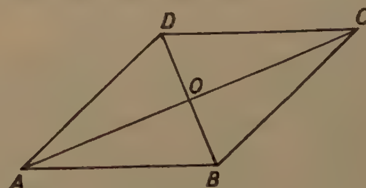


FIG. 14

(Note that all four angles at O are right-angles. It is sufficient to prove that one of them is a right-angle.)

From the similar triangles used in the last proof we could also deduce that $O\hat{A}B = O\hat{A}D$, i.e. AO bisects the angle BAD, thus proving the theorem: *The diagonals of a rhombus bisect the angles of the rhombus.*

It is worth noting that the diagonals of a parallelogram do *not* bisect the angles (of the parallelogram) *unless the parallelogram is a rhombus.*

"$ABCD$ is a parallelogram. $\therefore AC$ bisects the angle A," is an utterly wrong deduction and should never be written. But "$ABCD$ is a rhombus. $\therefore AC$ bisects the angle A," may be used at any time.

Since a square is a rectangle and also a rhombus, its diagonals are equal and they bisect the angles of the square.

A Special Kind of Quadrilateral

There is one other type of quadrilateral with a special property which will have great importance later. We begin by constructing a quadrilateral with this property and examine the other properties which it contains. Plot any three points A, B and C and the lines joining them (Fig. 15*a*). Draw any line from B to meet AC at O (Fig. 15*b*). Draw the line through A making an angle equal to CBO with AC. This line meets BO at D (Fig. 15*c*). Then $ABCD$ is a quadrilateral with the property that $C\hat{A}D = C\hat{B}D$. This is the special property we wish to discuss.

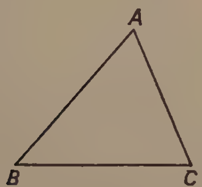


FIG. 15*a*

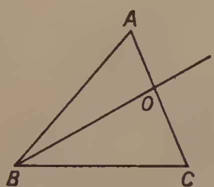


FIG. 15*b*

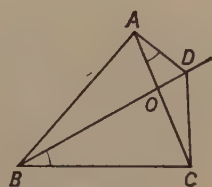


FIG. 15*c*

The angle CAD is called *the angle which CD subtends at A*; the angle CBD is *the angle which CD subtends at B*. (In general, the angle XKY is *the angle which XY subtends at K*.) Now CD is a side of the quadrilateral and AB is the opposite side, so that the property can be expressed in words as:

One side of the quadrilateral subtends equal angles at the ends of the opposite side.

A quadrilateral with this property is called, for reasons which will appear later, a CYCLIC quadrilateral. We now find that if a quadrilateral has this property for *one pair of opposite sides*, then *it has the same property for all the other pairs of opposite sides*. In drawing the quadrilateral $ABCD$ above, the given property is $C\hat{A}D = C\hat{B}D$. We shall now show that it also has the properties that $A\hat{C}D = A\hat{B}D$, $A\hat{C}B = A\hat{D}B$ and $C\hat{A}B = C\hat{D}B$. The first of these is easy to prove, the other two are rather harder.

Given. $C\hat{A}D = C\hat{B}D$.

To prove. $A\hat{C}B = A\hat{D}B$.

(Note that the given pair of equal angles are subtended by the side CD at the ends of the opposite side AB; the angles we shall prove equal are those subtended by the side AB at the ends of the opposite side CD.)

Proof. Let AC meet BD at O.

Then $A\hat{O}B = A\hat{D}O + D\hat{A}O = A\hat{D}B + C\hat{A}D$ ($A\hat{O}B$ is an exterior angle of $\triangle AOD$) and $A\hat{O}B = B\hat{C}O + C\hat{B}O = A\hat{C}B + C\hat{B}D$ ($A\hat{O}B$ is an exterior angle of $\triangle BOC$) $\therefore A\hat{C}B = A\hat{D}B$.

Given. $C\hat{A}D = C\hat{B}D$.

To prove. $C\hat{A}B = C\hat{D}B$, and $A\hat{B}D = A\hat{C}D$.

(Note that the given pair of equal angles are subtended by the side CD at the ends of the opposite side AB; the angles to be proved equal come from the *other pair* of opposite sides, BC and AD. The proof is much longer than the previous one.)

Proof. In the triangles AOD, BOC,

$O\hat{A}D = O\hat{B}C$ (given)

$A\hat{O}D = B\hat{O}C$ (vert. opp.)

$\therefore \triangle AOD \,|||\, \triangle BOC$

$$\therefore \frac{OA}{OD} = \frac{OB}{OC} \therefore OA.OC = OB.OD \therefore \frac{OA}{OB} = \frac{OD}{OC}$$

In the triangles AOB, DOC,

$\dfrac{OA}{OB} = \dfrac{OD}{OC}$ (proved)

$A\hat{O}B = D\hat{O}C$ (vert. opp.)

$\therefore \triangle AOB \,|||\, \triangle DOC$

$\therefore B\hat{A}O = C\hat{D}O$ and $A\hat{B}O = D\hat{C}O$, i.e. $C\hat{A}B = C\hat{D}B$ and $A\hat{B}D = A\hat{C}D$.

We notice also that in a quadrilateral with this special property there is also the special property that:

The product of the parts of one diagonal is equal to the product of the parts of the other diagonal, for in the proof just given we deduced that $OA.OC = OB.OD$.

We shall also see that a quadrilateral with this special property also has the property that a circle can be drawn to pass through its four vertices, i.e. if we draw the circle passing through three of the vertices, then this circle insists on passing through the fourth vertex also. It is from this property that the name CYCLIC QUADRILATERAL is taken.

It will help to understand these properties if a figure is drawn as follows: Draw a circle—one of about 2 in radius is recommended. Plot three points, A, B and C on the circle. Draw lines through B and C making equal angles with BA and CA (this can be done with a protractor). Whatever angle is selected as the size for the two equal angles, the two lines through B and C will be found to meet on the circle.

Problems with Cyclic Quadrilaterals

A cyclic quadrilateral, we have seen, has four pairs of equal angles, and this fact may be freely used in making calculations and deductions from given cyclic quadrilaterals. In these examples, note the way in which cyclic quadrilaterals are used, and the way in which the deductions are set out.

Example: $ABCD$ is a quadrilateral in which $C\hat{A}D = 26°$, $C\hat{B}D = 26°$, $A\hat{C}B = 74°$ and $B\hat{D}C = 38°$. Calculate $A\hat{B}D$.

Step 1. $B\hat{C}D + C\hat{B}D + B\hat{D}C = 180°$.

$\therefore B\hat{C}D + 26° + 38° = 180°$.

$\therefore B\hat{C}D = 116°$.

Step 2. $A\hat{C}D + A\hat{C}B = B\hat{C}D$.

$\therefore A\hat{C}D + 74° = 116°$. $\therefore A\hat{C}D = 42°$.

Step 3. $C\hat{A}D = C\hat{B}D$ (given). $\therefore ABCD$ is a cyclic quad.

$\therefore A\hat{B}D = A\hat{C}D = 42°$.

Answer: $A\hat{B}D = 42°$.

(Note that if the data were changed so that $C\hat{B}D = 25°$ instead of $26°$, we could not have taken Step 3, for the quadrilateral would not be cyclic. We could still calculate ACD as in Step 2, but it would not then be possible to calculate $A\hat{B}D$. Indeed, in this case, the angle ABD would not contain an exact number of degrees.)

Example: $ABCD$ is a quadrilateral in which $A\hat{D}B = A\hat{C}B$ and AC bisects the angle BAD. Prove that $B\hat{D}C = D\hat{B}C$.

Step 1. $A\hat{D}B = A\hat{C}B$ (given) $\therefore ABCD$ is cyclic.

Step 2. $ABCD$ is cyclic (proved) $\therefore B\hat{D}C = B\hat{A}C$ and $D\hat{B}C = D\hat{A}C$.

Step 3. $\therefore B\hat{D}C = B\hat{A}C = D\hat{A}C$ (given) $= D\hat{B}C$.

$\therefore B\hat{D}C = D\hat{B}C$.

Example: BD and CE are altitudes of the triangle ABC. Prove that $C\hat{E}D = C\hat{B}D$.

Step 1. $B\hat{D}C = B\hat{E}C$ (rt.-angles, given) $\therefore BEDC$ is cyclic.

Step 2. $\therefore C\hat{E}D = C\hat{B}D$.

Example: BD and CE are altitudes of the triangle ABC. The line through A, parallel to BC, meets BD produced at F. Prove that $A\hat{F}B = C\hat{E}D$.

Step 1. $B\hat{D}C = B\hat{E}C$ (rt.-angles given) $\therefore BEDC$ is cyclic.

Step 2. $\therefore C\hat{E}D = C\hat{B}D$.

Step 3. $AF \parallel BC$ (given) $\therefore A\hat{F}B = C\hat{B}D$

$\therefore A\hat{F}B = C\hat{E}D$.

We shall now prove two other results about cyclic quadrilaterals, which may then be freely used.

First, the opposite angles of a cyclic quadrilateral are supplementary.

Second, if the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

In the first, the given fact is that the quadrilateral is cyclic and we are required to deduce that the opposite angles are supplementary. In the second, the given fact is that the opposite angles are supplementary and we are required to deduce that the quadrilateral is cyclic. The proofs are very different in character.

Given. A cyclic quadrilateral $ABCD$.
To prove. $B\hat{A}D + B\hat{C}D = 2$ rt.-angles.
Proof. $B\hat{A}D = B\hat{A}C + C\hat{A}D = B\hat{D}C + C\hat{B}D$ since the quad. is cyclic.
But $B\hat{D}C + C\hat{B}D + B\hat{C}D = 2$ rt.-angles (angles of a triangle).
$\therefore BAD + BCD = 2$ rt.-angles.

Given: A quadrilateral $ABCD$ in which $B\hat{A}D + B\hat{C}D = 2$ rt.-angles.
To prove: $ABCD$ is cyclic.
Construction: (Under the heading "Construction" come points, not mentioned in "Given," which we need to prove the result. We need one more point here.)
AB and DC, produced, meet at F (Fig. 16).
Proof: $B\hat{C}F + B\hat{C}D = 2$ rt.-angles $= B\hat{A}D + B\hat{C}D$ (given).
$\therefore B\hat{C}F = B\hat{A}D.$
In the triangles FCB and FDA,
$\hat{F} = \hat{F},$
$F\hat{C}B = F\hat{A}D$ (proved)
$\therefore \triangle FCB \,|||\, \triangle FAD.$
$\therefore \dfrac{FC}{FB} = \dfrac{FA}{FD} \therefore \dfrac{FC}{FA} = \dfrac{FB}{FD}$
In the triangles FAC and FDB,
$\hat{F} = \hat{F},$
$\dfrac{FC}{FA} = \dfrac{FB}{FD}$
$\therefore \triangle FAC \,|||\, \triangle FDB.$
$\therefore F\hat{A}C = F\hat{D}B$, or $B\hat{A}C = B\hat{D}C.$
$\therefore ABCD$ is cyclic.

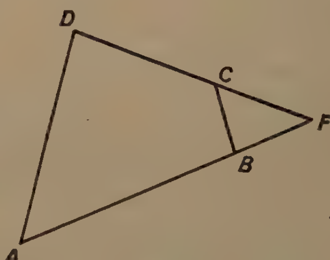


FIG. 16

EXERCISE 4D

1. $ABCD$ is a cyclic quadrilateral. The bisector of the angle BAC meets the bisector of the angle BDC at X. Prove that (a) quad. $ABXD$ is cyclic, (b) $AXCD$ is cyclic.

2. $ABCD$ is a cyclic quadrilateral and P is a point on BD produced. The line through P parallel to DC meets AC produced at Q. Prove that $ABQP$ is cyclic.

3. The altitudes, BD and CE, of a triangle ABC interesect at H. Prove that $ADEH$ is cyclic.

4. Continue Question 3 to prove that $D\hat{E}H = D\hat{A}H$.

5. Use the result of Question 4 and the result of worked Example 3 on page (Q16) to prove that $D\hat{A}H = C\hat{B}D$.

6. Use the result of Question 5 to prove that if AH produced meets BC at K, then $KBAD$ is cyclic, and $A\hat{K}B$ is a rt.-angle.

CHAPTER 5

DEDUCTIVE GEOMETRY

WHEN WE draw a geometrical diagram with certain given properties we find that there are certain other properties which the diagram contains, *and which we cannot prevent it from containing*. For example, if we draw a diagram of three points, A, B and C, with the given property that $AB = AC$, then it is impossible to draw such a diagram in which the angle ABC is not equal to the angle ACB, i.e. the diagram cannot have the property $AB = AC$ without also having the property $A\hat{B}C = A\hat{C}B$.

Again, if we draw a diagram with the properties $AB \parallel CD$ and $AD \parallel CB$, then it is impossible to draw such a diagram in which AB is not equal to CD, i.e. the diagram cannot have the properties $AB \parallel CD$ and $AD \parallel CB$ without also having the property $AB = CD$.

Such properties, which the diagram, so to speak, *insists on containing*, are called "deduced properties." In the first example above, $AB = AC$ is the given property and $A\hat{B}C = A\hat{C}B$ is the deduced property; in the second example, the given properties are $AB \parallel CD$ and $AD \parallel CB$; the deduced property is $AB = CD$. We have given only two examples here, but the remainder of this book will contain many more such examples and Deductive Geometry is the study of the properties which may be deduced from a set of given properties.

The given properties and the deduced properties are linked to each other by the sign $\therefore$, meaning "therefore." The two examples we have given would be written as (1) $AB = AC \ \therefore\ A\hat{B}C = A\hat{C}B$; (2) $AB \parallel CD$ and $AD \parallel CB$ $\therefore\ AB = CD$. The sign $\therefore$ is a very hard-working sign. When we write "Property X $\therefore$ Property Y" we are writing, in a very concise way, the long statement "It is impossible to draw a diagram *with* property X and *without* property Y."

Two sets of properties linked in this way are said to form a *deduction* and the second property (or set of properties) is said to be *deduced* from the first. A deduction can also be written using the words "if" and "then." Thus, the first of the above deductions may be written "If $AB = CD$, then $A\hat{B}C = A\hat{C}B$." And the second, "If $AB \parallel CD$ and $AD \parallel CB$, then $AB = CD$." Finally, if the properties can be given in words, without mentioning the names of particular points, then the deduction is called a THEOREM.

Thus, the two deductions above are the theorems, (i) if two sides of a triangle are equal, then the angles opposite those sides are equal; and (ii) if both pairs of opposite sides of a quadrilateral are parallel, then the opposite sides are equal. The statement can be made even shorter, for a quadrilateral with its opposite sides parallel is called a parallelogram, so we can rewrite (ii) above as: If a quadrilateral is a parallelogram, then its opposite sides are equal, or, shorter still: The opposite sides of a parallelogram are equal.

The *converse* of a deduction is obtained by interchanging the given properties with the deduced properties. Thus the converse of deduction (i) above is, $A\hat{B}C = A\hat{C}B \therefore AB = AC$, or, if two angles of a triangle are equal, then the sides opposite to those angles are equal. This statement is also found to be true, and it forms another theorem. But the converse of deduction (ii) is $AB = CD \therefore AB \parallel CD$ and $AD \parallel CB$, which is certainly not true, for it is not impossible to draw a diagram *with* the property $AB = CD$ and *without* the properties $AB \parallel CD$ and $AD \parallel CB$. Here we have a theorem which is true and its converse is false. We must, then, be on our guard against the mistake of thinking that because a theorem is true its converse is also true.

Deductive Geometry, then, investigates the properties of geometrical diagrams and the relations between these properties. From one theorem we may deduce others and we shall be concerned with the simpler theorems of geometry, namely, those which deal with straight lines and circles.

Now since our object is to deduce theorems from earlier theorems, there must be some theorems which we accept as true without deducing them from a previous one. These theorems are called *axioms*, and an axiom is a theorem which we assume to be true. We have made many assumptions in the earlier part of this book, but we are able to prove most of them and one of our objects is to have as few axioms as possible.

Congruence, Congruent Figures and Congruent Triangles

The idea of a carbon copy is a familiar one. A sheet of carbon paper is placed between two sheets of paper and any marks made on the upper sheet are copied exactly on the lower sheet. If we draw a geometrical diagram on the upper sheet we shall obtain an exact copy on the lower sheet. Two such diagrams are said to be *congruent*. The word congruent means fitting or agreeing and the relation between two such diagrams is called *congruence*. The two diagrams are still congruent if the sheets on which they are drawn are separated, and it is thus easy to think of two (or more) congruent diagrams in different places. The page of this book which you are now reading is congruent to the corresponding page of any other copy of this book. A square of side 4 inches is congruent to any other square of side 4 inches.

In drawing a geometrical diagram we plot points to fit given facts. From a set of given facts we find *either* that it is *impossible* to draw a diagram fitting the facts; *or* that *we can draw as many diagrams as we wish*, fitting the facts, i.e. if we are able to draw *one* diagram to fit the facts, then we can draw *as many as we wish*. Let us agree to call this diagram and all its copies *one congruent set*, or, more simply, *one set*. We now enquire how many sets of diagrams can be drawn from a given set of instructions.

***Example* 1:**

Given facts: $AB = 3$ in and $AB = 4$ in.

It is impossible to draw a diagram to fit these facts and we conclude that these facts give 0 sets of diagrams.

***Example* 2:**

Given facts: $AB = 4$ cm.

We can draw as many diagrams as we wish to fit these instructions, but they will all be congruent. We conclude that this fact gives 1 set of diagrams.

***Example* 3:**

Given facts: $AB = 4$ in and $BC = 3$ in.

The first of these facts ($AB = 4$ in) gives 1 set of diagrams, but C may be any point on the circle centre B radius 3 in. We conclude that these facts give *an unlimited number* of sets of diagrams.

We also observe that two given facts are not sufficient to limit the number of sets when the diagram contains 3 points, and we shall see in the next example that three given facts are necessary if we wish to reduce the number of sets to a limited number. Examples 4a, 4b and 4c each contain the same two given facts as Example 3, but each has one additional given fact.

***Example* 4a:**

Given facts: (1) $AB = 4$ in, (2) $BC = 3$ in and (3) $AC = 2$ in.

The first of these facts gives 1 (set of) diagram(s). We now have two facts to locate C. The locus of C from fact (2) is the circle centre B radius 3 in. The locus of C from fact (3) is the circle centre A radius 2 in. These two circles meet at two points, either of which is a possible position of C.

But the two triangles which we obtain are congruent (to each other) and we conclude that these three given facts lead to 1 set of diagrams.

***Example* 4b:**

Given facts: (1) $AB = 4$ in, (2) $BC = 3$ in and (3) $A\hat{B}C = 70°$.

The first of these facts leads to 1 diagram. The locus of C from fact (2) is the circle centre B, radius 3 in. The locus of C from fact (3) is the pair of lines from B, making angles of 70° with AB. Thus C has two possible positions, but they form two congruent triangles with A and B. We conclude that these three given facts lead to 1 set of diagrams.

***Example* 4c:**

Given facts: (1) $AB = 4$ in, (2) $BC = 3$ in and (3) $B\hat{A}C = 20°$.

The first of these facts gives 1 set of diagrams. The locus of C from fact (2) is the circle centre B, radius 3 in. The locus of C from fact (3) is the pair of lines from A making angles of 20° with AB. These two loci give four positions for C, but the four triangles obtained are not congruent. They give two pairs of congruent triangles, but a triangle of one pair is not congruent to a triangle of the other pair. We conclude that these three facts lead to 2 sets of diagrams.

From these three examples we begin to learn about *sets of facts from which we can deduce that two triangles are congruent.* From 4a it seems safe to assume that:

If two triangles agree about their three sides, then they are congruent, and we shall take this as an axiom. If two triangles are drawn with their sides in agreement, then they belong to the same set of congruent triangles, i.e. they are congruent (to each other).

From 4b it seems safe to assume that:

If two triangles agree about two sides and the angle which these two sides form, then the triangles are congruent, and we shall take this also as an axiom.

From 4c it is not safe to assume that if two triangles agree about two sides and an angle which is *not* formed by these two sides, then the triangles are congruent; for two triangles which agree in this manner may belong to two sets of triangles. It is also not safe to assume that two triangles agreeing in this manner are *not* congruent—for they *may* belong to the same set. In fact, it is not possible to make any deduction about congruence from the given facts that the triangles agree about two sides and an angle *not* formed by these two sides.

The word "included" is used to denote the angle formed by two sides of a triangle. The word must be used carefully, as must all geometrical words. By itself it means nothing, and it must contain the idea of being included between two sides. There is no such thing as "the included angle of a triangle," but we can use the word when it is clear which of the sides include the angle we wish to talk about.

In examples 4b and 4c above, the given facts concern two sides and an angle. In 4b, the angle is the one included by the sides; in 4c the angle is not the one included by the two sides. 4b, concerned with the included angle, leads to 1 set of congruent triangles and hence to an axiom about congruent triangles. 4c, on the other hand, concerned with the not-included angle, leads to 2 sets of congruent triangles, and we cannot derive an axiom from it.

We have now two axioms from which we can deduce that two triangles are congruent. There is also a third process:

***Example* 5:**

Given facts: $AB = 10$ cm, $B\hat{A}C = 40°$ and $A\hat{B}C = 60°$.

When AB has been drawn of length 10 cm, the locus of C from fact (2) is the pair of lines from A making 40° with AB; the locus of C from fact (3) is the pair of lines from B making 60° with AB. These two loci give two positions for C, but the two triangles obtained are congruent. We conclude that these facts lead to 1 set of triangles and our third axiom is:

If two triangles agree about a side and the angles at its ends, then the triangles are congruent.

These congruence axioms enable us to deduce facts about equal lines and angles (and areas, too) from given facts about equal lines and angles. But it is important to play fair, and not to make any *false* deductions.

A *false* deduction is one in which it *is possible* to draw a diagram *with* the given facts and *without* the deduced facts, e.g. it is a false deduction to say: $\triangle ABC$ is isosceles $\therefore A\hat{B}C = A\hat{C}B$. For it is possible to draw a triangle ABC with two equal sides and without $A\hat{B}C$ being equal to $A\hat{C}B$. Such a triangle is the one in which $AB = BC = 12$ cm and $A\hat{B}C = 1$ rt.-angle.

The deduction can be changed into a true deduction by saying: $AB = AC$. $\therefore A\hat{B}C = A\hat{C}B$, which has the same deduced fact as the false one, or by saying: ABC is isosceles $\therefore$ Two of the angles A, B and C are equal, which contains the same given fact as the false one.

You can avoid false deductions by observing these rules.

Rule 1: *There must be three given facts.*

The number three is important, for it is not possible to deduce congruence from two given facts; and if four facts are available, one of them is not needed and should be ignored, leaving the three required.

Rule 2: *The three facts must be agreements between the sides or angles of one triangle and the sides or angles of the other.*

The three facts will thus be three equations and it is good practice to reserve the left sides of these equations for parts of one triangle and the right sides for parts of the other triangle. It is also very good practice to use only the six letters which form the names of the two triangles, and not to allow any other letter at all among the given facts.

Rule 3: *The three given facts must be agreements between:*

either *the three sides* (of one triangle and the three sides of the other),
or *two sides and the included angle* (of one triangle, etc.),
or *one side and the angles at its ends* (of one triangle, etc.),
and if the three facts do not obey one of these three rules, then a false deduction is inevitable.

There is one case of congruence which does not obey Rule 3 and yet leads to congruent triangles. This is the case in which two right-angled triangles agree about their hypotenuses, and about one other side. (The hypotenuse of a right-angled triangle is the side opposite to the right-angle.) This case breaks Rule 3, but the triangles are nevertheless congruent, and we have a special case applying to *right-angled triangles only*.

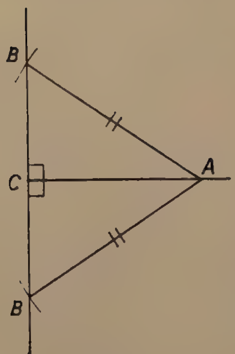


FIG. 1

If we construct a right-angled triangle ABC given the hypotenuse AB and one of the other sides AC, we begin by drawing AC. This gives 1 set of diagrams. We now draw a right-angle at C and it remains to locate the point B. The locus of B from the right-angle fact is the perpendicular through C to AC, and the locus of B, from the fact that the hypotenuse is to be a given length, is a circle with centre A. These two loci intersect at 2 points, so there are 2 positions for B (Fig. 1). But the two positions lead to congruent triangles, so that these data give only 1 set of triangles.

We conclude that:

Two right-angled triangles which agree about their hypotenuses and about one other side are congruent.

When using this theorem, care should be taken to mention the three facts

(i) that the triangles *are* right-angled,

(ii) they agree about their hypotenuses, and

(iii) they agree about another side.

Example:

Given: AB and BC are two equal and perpendicular lines. X is a point in AB. The circle centre A with radius CX meets BC at Y.

To prove: That $BX = BY$.

Proof: In the rt.-angled triangles ABY and CBX,

hyp. $AY =$ hyp. CX (given)

$AB = CB$ (given)

$\therefore \triangle ABY \equiv \triangle CBX \therefore BX = BY$.

It would be *wrong* to say:

In the triangles ABY and CBX,

$AY = CX$ (given)

$AB = CB$ (given)

$A\hat{B}Y = C\hat{B}X$ (rt.-angles)

$\therefore \triangle ABY \equiv \triangle CBX$,

for the angle is not the included angle and we have failed to prove that the triangles are congruent.

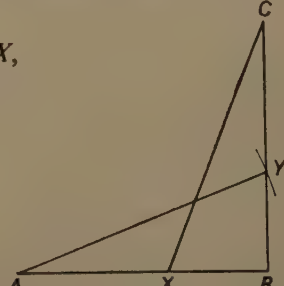


FIG. 2

EXERCISE 5A

1. AB and CD bisect each other at O. Prove that AC is parallel to BD.
Draw, freehand, a line AB; mark O, its mid-point; through O draw another line and mark C and D at equal distances from O; join AC and BD.
Prove the triangles AOC and BOD congruent.
Deduce two equal angles from which it can be deduced that

AC is parallel to BD.

Following these instructions, we write:

In the $\triangle$s AOC, BOD,

$$AO = OB$$
$$CO = OD$$
$$A\hat{O}C = B\hat{O}D \text{ (vert. opp.)}$$
$$\therefore \triangle AOC \equiv \triangle BOD$$
$$\therefore A\hat{C}O = O\hat{D}B\text{, and these are alternate.}$$
$$\therefore AC \text{ is } \parallel \text{ to } BD.$$

2. $ABCD$ is a square, and M is the mid-point of AB. Prove that $MC = MD$.

3. AX bisects the angle BAC and $A\hat{X}B = A\hat{X}C$. Prove that $AB = AC$.

4. $ABCD$ is a quadrilateral in which $AB = AD$ and $CB = CD$. Prove that AC bisects the angles BAD and BCD.

5. M is the mid-point of AB, and MX is perpendicular to AB. Prove that $XA = XB$.

6. $ABCD$ is a quadrilateral in which $AB = CD$ and $AD = BC$. Prove that AB is parallel to DC.

7. X and Y are points on the bisector of the angle BAC such that $AX = AB$ and $AY = AC$. Prove that $BY = CX$. Prove also that $A\hat{B}Y = A\hat{X}C$.

8. ABC is a triangle in which $AB = AC$. The bisector of the angle BAC meets BC at D. Prove that $\triangle ABD \equiv \triangle ACD$ and that $\hat{B} = \hat{C}$.

9. $ABCD$ is a quadrilateral in which AB is equal and parallel to DC. Prove that AD is parallel to BC.

10. X is a point in the line AB. PAX and QBX are equilateral triangles. Prove that $\triangle AXQ \equiv \triangle PXB$. Deduce that $AQ = BP$.

11. $AB = AC$. P, Q are points in AB, AC such that $AP = AQ$. Prove that $\triangle AQB \equiv \triangle APC$ and deduce that $BQ = CP$.

12. $ABCD$ is a square. A circle with centre A cuts BC at P and DC at Q. Prove that $\triangle ABP \equiv \triangle ADQ$.

13. In the figure of Question 12, prove that AC bisects the angle PAQ.

CHAPTER 6

STATEMENT, APPLICATION AND PROOF OF A THEOREM

HAVING ACCEPTED the three congruence theorems we can go on to establish other theorems. We have already established two theorems in the preceding set of exercises. No. 8 contains the given fact that $AB = AC$ and the deduced fact is that $A\hat{B}C = A\hat{C}B$, giving us the theorem proving that:

If two sides of a triangle are equal, the angles opposite to those sides are equal.

In proving No. 9 we have proved the theorem that:

If a quadrilateral has one pair of sides equal and parallel, the other two sides are parallel.

When we have obtained a theorem there are three important things which we can do with it. First we can *state* it. Each theorem can be stated in many different ways, but the statement must always make it clear what are the given properties of the figure and what are the deduced properties. In the first of the two theorems above, the given fact is that two sides of a triangle are equal; the deduced fact is that two angles are equal. In the second, the given facts are that one pair of sides are both equal and parallel; the deduced fact is that the other pair is parallel. The given facts are often indicated by the word "if" which precedes them, but the statement of a theorem need not contain this word, e.g. the first theorem above may be stated as:

The angles at the base of an isosceles triangle are equal.

On reading this statement we have to realize that the given fact is that the triangle is isosceles (i.e. two of its sides are equal) and that the deduced fact is that the base angles are equal.

Secondly we can *prove* a theorem, i.e. we can deduce its truth from preceding theorems.

Thirdly—and most important—we can *apply* the theorem, i.e. we can use it to make deductions about a diagram which is known to contain the given facts in the theorem.

A theorem can be applied even if its proof is not known. If the statement of the theorem is known—and understood—then the theorem is ready to be applied to any suitable figure. We shall give statements of theorems, ready to be applied, and we shall frequently leave the proof to a later period.

How to Apply a Theorem

A theorem is a general statement that all diagrams which contain certain given facts also contain certain deduced facts. The statement is in general terms and does not refer to any point, line, etc. *by name*.

When we apply a theorem to a particular diagram, we first *state* that the diagram contains all the given facts of the theorem and then (after the sign $\therefore$) we *state* the deduced facts (or such of them as we require). These statements are all made *in terms of the letters of the diagram*. We now give examples of the application of theorems to diagrams.

1. In Fig. 1, $AB = AC$ and $BX = BC$. *To prove:* That $A\hat{B}C = B\hat{X}C$.

Theorem	*Application*
If two sides of a triangle are equal, the opposite angles are equal.	$AB = AC$
	$\therefore A\hat{C}B = A\hat{B}C$
	$BX = BC$
	$\therefore B\hat{C}X = B\hat{X}C$
	$\therefore A\hat{B}C = A\hat{C}B = B\hat{C}X = B\hat{X}C$
	$\therefore A\hat{B}C = B\hat{X}C$

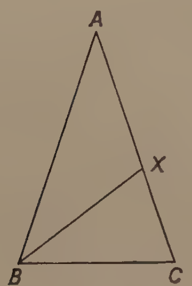


FIG. 1

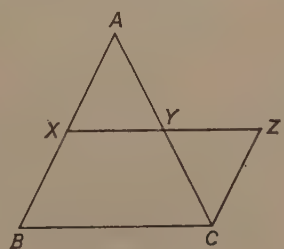


FIG. 2

2. In Fig. 2, Y is the mid-point of AC, XYZ is parallel to BC and CZ is parallel to AX.

To prove: That $AX = XB$.

Theorem	*Application*
If two triangles agree about one side and the angles at the ends of these sides	In the $\triangle$s AXY, CZY
	$AY = CY$ (given)
	$A\hat{Y}X = C\hat{Y}Z$ (vert. opp.)
	$X\hat{A}Y = Z\hat{C}Y$ ($AX \parallel CZ$)
then the triangles are congruent.	$\therefore \triangle AXY \equiv \triangle CZY$
	$\therefore AX = ZC$
If both pairs of opposite sides of a quadrilateral are parallel (i.e. if the quad. is a $\parallel$gm.)	$XZ \parallel BC$, $XB \parallel ZC$
	$\therefore BXZC$ is a $\parallel$gm.
then its opposite sides are equal	$\therefore ZC = XB$
	$\therefore AX = ZC = XB$, or $AX = XB$.

3. In Fig. 3, $AB = AC$ and $XB = XC$.

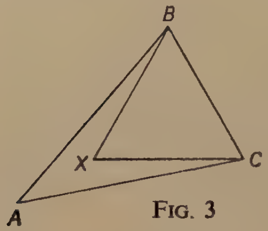


FIG. 3

To prove: That $A\hat{B}X = A\hat{C}X$.

$AB = AC \therefore A\hat{B}C = A\hat{C}B$ (*1st application of isos.* $\triangle$ *theorem*)

$XB = XC \therefore X\hat{B}C = X\hat{C}B$ (*2nd application*)

$\therefore A\hat{B}C - X\hat{B}C = A\hat{C}B - X\hat{C}B$,

or $A\hat{B}X = A\hat{C}X$

In the foregoing examples it is not necessary to put in the words in italics; they are to explain the theorems which we apply at each step.

EXERCISE 6A

1. In Fig. 4, $AB = AC$ and EAF is parallel to BC. Apply the isosceles triangle theorem to ABC and use the deduced fact to prove that $E\hat{A}B = F\hat{A}C$.

2. In Fig. 5, $AB = AC$, AX is parallel to BC and $XA = XC$. Prove that $X\hat{C}A = A\hat{B}C$. (In this example, apply the isosceles triangle theorem and also the theorem: Alternate angles between parallel lines are equal.)

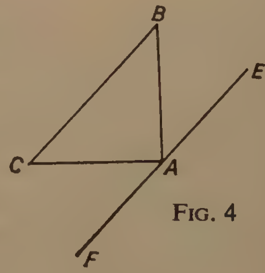


FIG. 4

The converse of the Isosceles Triangle Theorem is: If two angles of a triangle are equal, then the sides opposite to those angles are equal. Apply this theorem in the following exercises.

3. In Fig. 6, $\hat{B} = \hat{C}$.
State the deduced fact.

4. In Fig. 7, AE is parallel to BC and AE bisects $D\hat{A}C$. Prove that $AB = AC$.

5. In Fig. 8, $AB = AC$ and XB, XC are the bisectors of the angles ABC and ACB. Prove that $XB = XC$.

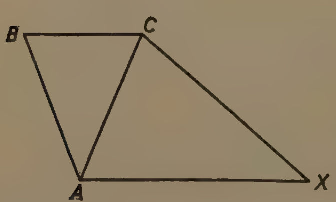


FIG. 5

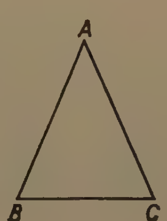


FIG. 6

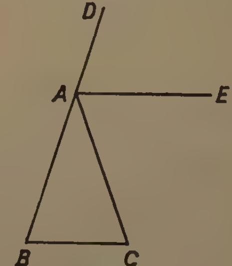


FIG. 7

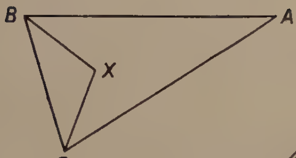


FIG. 8

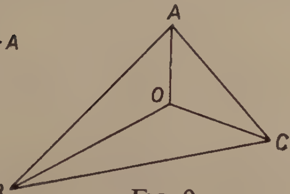


FIG. 9

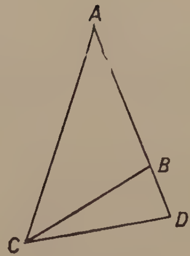


FIG. 9a

6. In Fig. 9, $O\hat{A}B = O\hat{B}A$ and $O\hat{A}C = O\hat{C}A$. Prove that $O\hat{B}C = O\hat{C}B$.

7. In Fig. 9a, $AB = BC = CD$ and $\hat{A} = 40°$. Calculate $A\hat{C}D$.

Proof of the Isosceles Triangle Theorem

Given: $AB = AC$

To prove: $A\hat{B}C = A\hat{C}B$

Construction: M is the point in which the bisector of $B\hat{A}C$ meets BC (Fig. 10).

Proof: In the $\triangle$s ABM and ACM

$AB = AC$ (given)

$AM = AM$

$B\hat{A}M = C\hat{A}M$ (construction)

$\therefore \triangle ABM \equiv \triangle ACM$

$\therefore A\hat{B}M = A\hat{C}M$, i.e. $A\hat{B}C = A\hat{C}B$

FIG. 10

Proof of the Converse of the Isosceles Triangle Theorem.

Given: $A\hat{B}C = A\hat{C}B$

To prove: $AB = AC$

Construction: M is the point in which the bisector of $B\hat{A}C$ meets BC (Fig. 10a).

Proof: In the $\triangle$s ABM and ACM

$AM = AM$

$B\hat{A}M = C\hat{A}M$ (construction)

$A\hat{B}M = A\hat{C}M$ (given)

$\therefore \triangle ABM \equiv \triangle ACM$

$\therefore AB = AC$

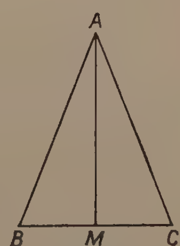


FIG. 10a

It should be noted that although the diagrams used in proving these two theorems are the same, the given facts about them are different in the two cases. The diagram is the least important part of the proof. The given facts in the diagram are what really matter.

Note also the heading "Construction." This heading is used for naming points which are required in the proof, but which have no part in the given

facts or in the facts which we wish to deduce. The heading Construction should be reserved for naming new points of this kind, and it is unnecessary to include in this heading instructions such as "Join XY" (if X and Y are already in the diagram).

From the congruent triangles in the above proofs we could also deduce two facts about M. First, that M is the mid-point of BC (for $BM = MC$ from the congruent triangles), and secondly that AM is perpendicular to BC (for $A\hat{M}B = A\hat{M}C$).

The Mid-point Theorems

1. The line joining the mid-points of two sides of a triangle is parallel to the third side. It is also half the third side. In other words, if two points are known to be mid-points of two sides of a triangle, then the line joining them is parallel to the third side and is equal to half of it.

This theorem can be applied to any diagram which contains the mid-points of two lines which start at the same point, e.g. if X is known to be the mid-point of AB and Y is known to be the mid-point of AC we may say at once that XY is parallel to BC. We may also say that XY is half of BC. The argument looks like this: X is the mid-point of AB; Y is the mid-point of AC $\therefore$ $XY \parallel BC$ and/or $XY = \frac{1}{2}BC$.

2. The line through the mid-point of one side of a triangle parallel to a second side bisects the third side. In other words, if a line passes through the mid-point of one side and is parallel to a second, then it passes through the mid-point of the third side. In this case, the given facts are (i) the line passes through a mid-point, (ii) it is parallel to a second side. The deduced fact is that it passes through the mid-point of the third side. The argument looks like this: X is the mid-point of AB; XY is parallel to BC; $\therefore$ Y is the mid-point of AC (or $AY = YC$ if preferred).

EXERCISE 6B

Apply these theorems to diagrams with the following given facts:

1. H is the mid-point of XY. K is the mid-point of YZ.

2. L is the mid-point of AB. M is the mid-point of AC. N is the mid-point of AD.

3. X is the mid-point of AB. Y is the mid-point of BC. Z is the mid-point of CD. W is the mid-point of DA.

4. In the diagram of No. 3, prove (i) that XY is parallel to WZ; (ii) that $XY = WZ$.

5. In the diagram of No. 3 find two other lines which are equal and parallel.

6. $ABCD$ is a quadrilateral and K is the mid-point of AB. The line through K parallel to BC meets the diagonal AC at L. The line through L parallel to CD meets AD at M. Prove that L is the mid-point of AC. Use this deduced fact to prove that M is the mid-point of AD. Use the known facts to prove that KM is parallel to BD.

7. ABC is a triangle in which $AB = AC$. O is any point inside the triangle. K, L and M are the mid-points of OA, OB, OC respectively. Prove that $KL = KM$.

8. ABC is a triangle. X is the mid-point of AB and Y the mid-point of AC. T is any point in BC. AT and XY meet at M. Prove that M is the mid-point of AT.

9. ABC is a triangle. F is the mid-point of AB and E the mid-point of AC. BE and CF meet at G. AG is produced to K, such thát $AG = GK$. By applying the mid-point theorem to the triangles ABK and ACK prove that $BGCK$ is a parallelogram (i.e. that $BG \parallel KC$ and $CG \parallel KB$).

(The proof of the mid-point theorems will be given later.)

Theorems on Parallelograms

These theorems are of two kinds. The first kind tells us the deductions we may make from a figure known to be a parallelogram; the second kind tells us what facts we require to know before we can say that a quadrilateral is a parallelogram.

1. If $ABCD$ is a parallelogram any (or all) of the following facts may be deduced.

(a) $AB \parallel CD$ and $AD \parallel BC$

(b) $AB = CD$ and $AD = BC$

(c) $\hat{A} = \hat{C}$ and $\hat{B} = \hat{D}$.

(d) AC bisects BD (i.e. if AC meets BD at K, then $BK = KD$) and BD bisects AC.

Note that all these facts are *deduced* from the given fact that $ABCD$ is a parallelogram.

These theorems may be stated as follows:

(a) The opposite sides of a parallelogram are parallel. (This is not really a theorem; it is a property which a quadrilateral must have before it can be called a parallelogram. But this does not prevent our deducing these facts if they are needed.)

(b) The opposite sides of a parallelogram are equal, i.e. if a quadrilateral is a parallelogram, then its opposite sides are equal. This theorem has already been proved, but we shall give another proof later.

(c) The opposite angles of a parallelogram are equal.

(d) The diagonals of a parallelogram bisect one another.

2. What facts are required before we may say that $ABCD$ is a parallelogram? Two facts (about the sides of the quadrilateral which is being considered) are necessary before we may say that the quadrilateral is a parallelogram.

(a) $AB \parallel DC$ and $AD \parallel BC$ $\quad\therefore ABCD$ is a $\parallel$ gm.
(b) $AB \parallel DC$ and $AB = DC$ $\quad\therefore ABCD$ is a $\parallel$ gm.
(c) $AD \parallel BC$ and $AD = BC$ $\quad\therefore ABCD$ is a $\parallel$ gm.
(d) $AB = CD$ and $AD = BC$ $\quad\therefore ABCD$ is a $\parallel$ gm.

These theorems may be stated as follows:
(a) If the opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram. (This, too, is not really a theorem. It says that if a quadrilateral has the properties of a parallelogram then it may be called a parallelogram! Nevertheless it is very often used and should be noted.)
(b) and (c) If a quadrilateral has one pair of sides equal and parallel, then it is a parallelogram.
(d) If a quadrilateral has both pairs of opposite sides equal, then it is a parallelogram.

(b) and (c) should be studied very carefully. Note that in each case it is the same pair of sides which is both equal and parallel. A theorem, you may remember, is a statement that it is *impossible* to draw a diagram which *has* the given properties *and which does not have* the deduced ones.

Example: It is impossible to draw a diagram $ABCD$ in which $AD \parallel BC$ and $AD = BC$ and *which is not a parallelogram*. (This is theorem (c) above.) But it is not impossible to draw a diagram $ABCD$ in which $AB \parallel CD$ and $AD = BC$ and *which is not a* parallelogram. (Fig. 11 shows such a diagram.)

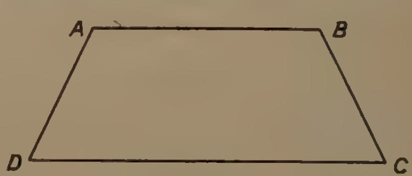


FIG. 11

Before making a deduction like (b) or (c) above, make sure that the same pair of sides is mentioned in both statements, and if, unfortunately, your given facts are $AB \parallel DC$ and $AD = BC$, then you *cannot* deduce that $ABCD$ is a $\parallel$ gm. and you will have to try something else.

EXERCISE 6C

1. $ABCD$ is a parallelogram. G is the mid-point of AB and H is the mid-point of CD. Apply theorem 1 (b), to prove that $AG = HC$. Now apply theorem 2 (b) to prove that $AGCH$ is a parallelogram. Now apply theorem 1 (a) to prove that AH is parallel to GC.
(This example could be set as one problem: $ABCD$ is a parallelogram. G is the mid-point of AB and H is the mid-point of CD. Prove that AH is parallel

to GC. This problem would be solved by going through the three steps shown.)

2. M is the mid-point of AB and $AMXY$ is a parallelogram. Prove that MY is parallel to BX. (Apply 1 (b) to $AMXY$ and deduce that $BM = YX$. Apply 2 (b) to $MBXY$ and then apply 1 (a) to $MBXY$.)

3. Draw any triangle. Through each vertex draw the line parallel to the opposite side. These three lines form a second triangle. Prove that each vertex of the first triangle is the mid-point of a side of the second triangle. (First letter your diagram with any letters you like.)

4. M is the mid-point of the line AC and M is also the mid-point of the line BD. Prove that the triangles AMB and CMD are congruent. Deduce two facts from this congruence which, by applying theorem 2 (b) above, will prove that $ABCD$ is a parallelogram.

5. $ABPQ$ is a parallelogram. $ABRS$ is also a parallelogram. Prove that $PQSR$ is a parallelogram. (Apply 1 (a), 1 (b) and 2 (b).)

6. $ABCD$ is a parallelogram. The line through D parallel to the diagonal AC meets BA produced at X and BC produced at Y. By applying 2 (a) to $ACDX$ and $ACYD$, and 1 (b) to two parallelograms, deduce that D is the mid-point of XY.

7. X is the mid-point of AB. Y is the mid-point of AC. Y is also the mid-point of XZ. Prove:

(i) that the triangles AXY and CZY are congruent;
(ii) that AX is equal and parallel to ZC;
(iii) that $BXZC$ is a parallelogram;
(iv) that XY is parallel to BC and that $XY = \frac{1}{2}BC$.

(*N.B.*—This is one of the mid-point theorems, that the line joining the mid-points (X and Y) of two sides of a triangle (AB and AC) is parallel to the third side (BC) and equal to half of it. The point Z plays no part in the given facts or the deduced facts. It is a "construction" point.)

Proofs of the Parallelogram Theorems

1 (a) requires no proof; it follows from the definition of a parallelogram

1 (b) *Given:* $ABCD$ is a parallelogram.

To prove: $AB = CD$ and $AD = BC$.

Proof: In the triangles ABC and CDA,

$AC = AC$

$B\hat{A}C = D\hat{C}A$ ($AB \parallel DC$, given)

$B\hat{C}A = D\hat{A}C$ ($BC \parallel AD$, given)

$\therefore \triangle ABC \equiv \triangle CDA$

$\therefore AB = CD$ and $AD = BC$

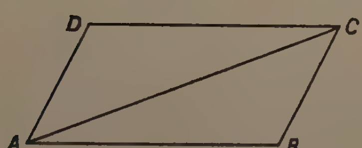


FIG. 12

1 (c) *Given:* $ABCD$ is a parallelogram.
To prove: $\hat{A} = \hat{C}$ and $\hat{B} = \hat{D}$.
Proof: Repeat the proof of 1 (b) as far as $\triangle ABC \equiv \triangle CDA$, then:
$\triangle ABC \equiv \triangle CDA$
$\therefore \hat{B} = \hat{D}$
From the congruent triangles ABD and CDB we could similarly deduce that $\hat{A} = \hat{C}$ $\therefore$ $\hat{A} = \hat{C}$ and $\hat{B} = \hat{D}$.

1 (d) *Given:* $ABCD$ is a parallelogram whose diagonals, AC and BD, meet at K (Fig. 13).
To prove: K is the mid-point of AC and also of BD.
Proof: In the triangles AKB and CKD

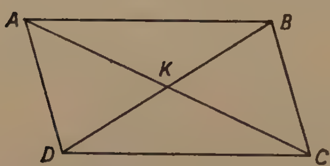


FIG. 13

$AB = CD$ (opp. sides of a $\|$gm.)
$K\hat{A}B = K\hat{C}D$ ($AB \parallel CD$)
$K\hat{B}A = K\hat{D}C$ ($AB \parallel CD$)
$\therefore \triangle AKB \equiv \triangle CKD$
$\therefore AK = KC$ and $BK = KD$
i.e. K is the mid-point of AC and also of BD.

2 (a) requires no proof. If a quadrilateral has the special properties that its opposite sides are parallel, then it is a parallelogram by definition.

2 (b) *Given:* $AB \parallel DC$ and $AB = DC$.
To prove: $ABCD$ is a parallelogram.
Proof: In the triangles ABC and CDA
$AB = DC$
$C\hat{A}B = A\hat{C}D$ ($AB \parallel CD$)
$AC = AC$
$\therefore \triangle ABC \equiv \triangle CDA$
$\therefore A\hat{C}B = C\hat{A}D$, and these are alternate angles.
$\therefore BC \parallel AD$
In $ABCD$, $AB \parallel DC$ (given) and $BC \parallel AD$ (proved)
$\therefore ABCD$ is a parallelogram.

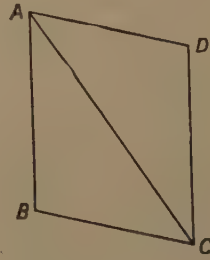


FIG. 14

2 (c) has exactly the same statement as 2 (b), so the proof is the same.

2 (d) *Given:* $AB = CD$ and $AD = BC$.
To prove: $ABCD$ is a parallelogram.
Proof: In the triangles ABC and CDA
$AB = CD$
$BC = DA$

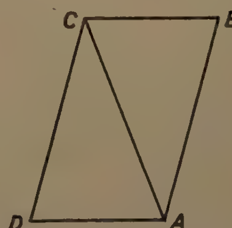


FIG. 14*a*

$AC = AC$
$\therefore \triangle ABC \equiv \triangle CDA$
$\therefore B\hat{A}C = D\hat{C}A$ and these are alternate $\therefore AB \parallel CD$
and since $\triangle ABC \equiv \triangle CDA$, $A\hat{C}B = C\hat{A}D$ and these are alternate
$\therefore BC \parallel AD.$
In $ABCD$, $AB \parallel CD$ (proved) and $BC \parallel AD$ (proved)
$\therefore ABCD$ is a parallelogram.

Note that all these proofs have exactly the same diagram, but the given facts are different in each case. The triangles are proved congruent from the given facts.

Proofs of the Mid-point Theorems

1. The line through the mid-point of one side of a triangle parallel to another side bisects the third side. (The given facts about the line are (i) that it passes through the mid-point of one side of the triangle, (ii) that it is parallel to another side. The deduced fact is that it bisects the third side.)

Given: M is the mid-point of AC. MN is the line through M parallel to CB, meeting AB at N.

To prove: N is the mid-point of AB.

Construction: L is the point in which the line through M, parallel to AB, meets BC.

Proof: In the triangles ANM, MLC
$AM = MC$ (given)
$N\hat{A}M = L\hat{M}C$
($AN \parallel ML$, construction)
$A\hat{M}N = M\hat{C}L$ ($MN \parallel CL$, given)
$\therefore \triangle ANM \equiv \triangle MLC$
$\therefore \quad AN = ML$
In the quad. $BLMN$
$BL \parallel NM$ (given)
and $BN \parallel LM$ (construction)
$\therefore BLMN$ is a parallelogram
$\therefore BN = LM$
$\therefore$ as $LM = AN$ (proved) $BN = AN$
i.e. N is the mid-point of AB.

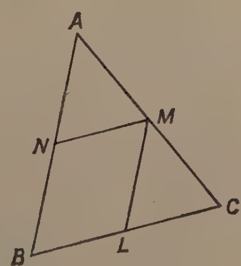


FIG. 15

A note on the Given Facts: No mention is made of the positions of A, B or C. Thus they can be plotted anywhere. M is now located as the mid-point of AC and the remaining point N is located by the two facts (i) MN is parallel to CB and N is a point on AB.

A note on Construction: This is to introduce a new point, L, which is necessary for the proof, but which is not needed among the given facts L is located by the two facts (i) L lies in BC and (ii) LM is parallel to AB.

2. The line joining the mid-points of two sides of a triangle is parallel to the third side. (The given facts about the line are (i) that it passes through the mid-point of one side and (ii) that it passes through the mid-point of a second side. The deduced fact is that it is parallel to the third side.)

Given: M is the mid-point of AC and N is the mid-point of AB.

To prove: NM is parallel to BC.

Construction: Produce NM to X such that $MX = NM$.

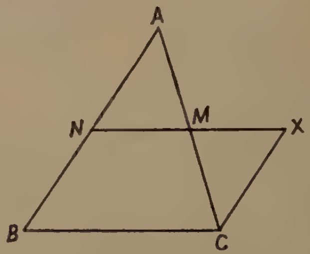


FIG. 16

Proof: In the triangles ANM, CXM

$AM = MC$ (given)

$NM = MX$ (construction)

$A\hat{M}N = C\hat{M}X$ (vert. opp.)

$\therefore \triangle ANM \equiv \triangle CXM$

$\therefore AN = CX \ldots$(i) and $N\hat{A}M = X\hat{C}M \ldots$(ii)

From (i) and $AN = NB$ (given), $CX = NB \ldots$(iii)

From (ii) $AN \parallel XC$, i.e. $CX \parallel NB \ldots$(iv)

From (iii) and (iv) $CXNB$ is a parallelogram

$\therefore NX \parallel BC$, i.e. NM is parallel to BC.

3. The line joining the mid-points of two sides of a triangle is half the third side.

We shall not give a complete proof here, for it is very like Proof 2.

Given: As in 2.

To prove: $NM = \frac{1}{2}BC$.

Construction: As in 2.

Proof: As in 2 as far as $CXNB$ is a parallelogram

$\therefore NX = BC$, but $NX = 2NM$ (construction)

$\therefore 2NM = BC$ or $NM = \frac{1}{2}BC$.

The Equal Intercept Theorem

When a straight line meets two other lines, the part of it which lies between the two points of intersection is referred to as "the intercept on this line made by the other two." Thus an intercept is made *by* two lines *on* a third line, and the word has no meaning unless it is clear which two lines *make* the intercept, and which line they *make* it *on*.

We may, then, speak of "the intercept made on XY by AB and CD," but not of "the intercept made by AB and CD," or "the intercept on XY."

When a line is met by several lines, we reserve the word "intercept" for those parts of the line between two intersections which are not separated by another intersection.

Thus, in Fig. 17, the "intercepts" *on* the line XY *made by* the other four lines are AC, CD and DB. AD will not be called an intercept (on XY made by the four lines) as there is another intersection (C) between A and D.

It will thus be seen that the number of intercepts on a line made by a number of other lines is one less than the number of lines making the intercepts. Seven lines would make six intercepts on another line and n lines would make $(n - 1)$ intercepts.

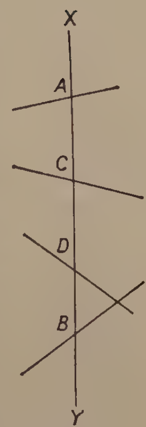


FIG. 17

We will now draw a set of parallel lines which make equal intercepts on a given line. To do this we draw the given line and mark off along it points at equal intervals. Through the ends of these intervals we draw parallel lines (Fig. 18).

The Equal Intercept Theorem states that if another line is now drawn to meet these parallels, they will make equal intercepts on this line also.

Fig. 18 shows equal intercepts marked with compasses on one line, and parallels drawn through the ends of the intercepts. The intercepts on the other line are found to be equal. You should practise doing this with compasses and set-square, and verify (with compasses) that the second set of intercepts are equal (to one another).

The statement of the theorem, then, is:
A set of parallel lines which make equal intercepts on one line will make equal intercepts on any other line.

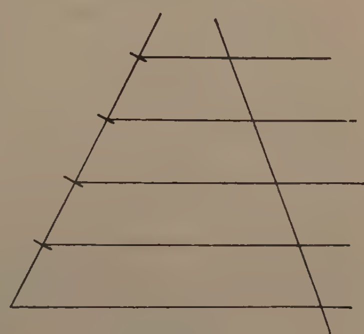


FIG. 18

We can now apply this theorem to the problem of dividing a given line into a given number of parts.

Let AB be the given line and the given number of parts 5 (Fig. 19).

Method: Draw any line through A.

Mark off along this line 5 equal distances, starting from A. Let these distances be

AX_1, X_1X_2, X_2X_3, X_3X_4, X_4X_5.

Join X_5B.

Through X_1 X_2 X_3 X_4 draw lines parallel to X_5B and let these lines meet AB at Y_1 Y_2 Y_3 Y_4.

Then these four points divide AB into 5 equal parts.

The proof is obtained by applying the equal intercept theorem.

Proof: X_1Y_1, X_2Y_2, X_3Y_3, X_4Y_4, X_5B are parallel.

The intercepts on AX_5 (AX_1, X_1X_2, X_2X_3, X_3X_4, X_4X_5) are equal.

$\therefore$ The intercepts on AB (AY_1, Y_1Y_2, Y_2Y_3, Y_3Y_4, Y_4B) are equal, i.e. the line AB has been divided into 5 equal parts.

In practice, the equal distances AX_1, etc. are marked with compasses, and any convenient distance may be chosen. The parallels are drawn with a set-square.

A more convenient method of constructing these points, Y, requires only two parallels and is as follows:

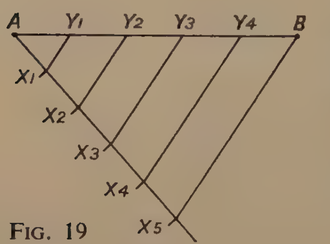


FIG. 19

Through A draw any line and mark off five points, X, as before.

Through B draw the line parallel to AX and mark off 5 distances

$$BZ_1,\quad Z_1Z_2,\quad Z_2Z_3,\quad Z_3Z_4,\quad Z_4Z_5,$$

each equal to AX_1. Then the lines

$$X_1Z_4,\quad X_2Z_3,\quad X_3Z_2,\quad X_4Z_1$$

will divide AB into 5 equal parts.

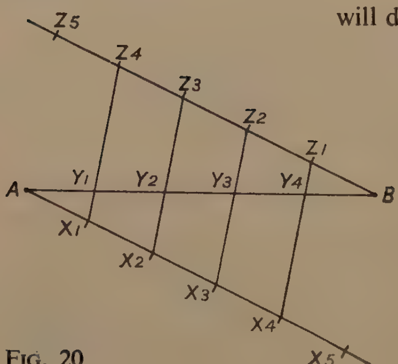


FIG. 20

Proof: $AX_1 = Z_5Z_4$ and $AX_1 \parallel Z_5Z_4$.

$\therefore$ $AX_1Z_4Z_5$ is a parallelogram.

$\therefore$ $AZ_5 \parallel X_1Z_4$.

Similarly $X_1Z_4 \parallel X_2Z_3$, and all the lines joining the X's and Z's are parallel.

Also, the intercepts on AX_5 are equal. $\therefore$ The intercepts on AB are equal, i.e. the line AB has been divided into 5 equal parts.

Proof of the Equal Intercept Theorem

Given: Three parallel lines meeting a line at A, B and C such that $AB = BC$, and meeting another line at X, Y and Z.

To prove: $XY = YZ$.

Construction: The line through X parallel to AB meets BY at P and the line through Y parallel to BC meets CZ at Q.

Proof: In $AXPB$, $AX \parallel PB$ (given) and $AB \parallel XP$ (construction)

$\therefore$ $AXPB$ is a parallelogram $\therefore$ $AB = XP$

Similarly $BC = YQ$. But $AB = BC$ (given)

$\therefore$ $XP = YQ$.

In the triangles XPY, YQZ

$$\begin{aligned} XP &= YQ \text{ (proved)} \\ P\hat{X}Y &= Q\hat{Y}Z \ (XP \parallel AC \parallel YQ) \\ X\hat{P}Y &= P\hat{Y}Q \ (XP \parallel YQ) \\ &= Y\hat{Q}Z \ (PY \parallel QZ) \\ \therefore \triangle XPY &\equiv \triangle YQZ \\ \therefore XY &= YZ. \end{aligned}$$

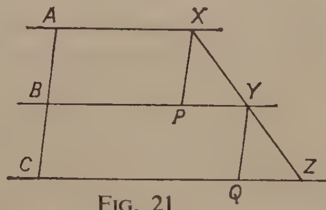


FIG. 21

Note that we have proved the theorem for three parallels only. A similar proof would apply for any number of parallels.

The Equal Intercept Theorem can be applied only to a figure in which there are a number of parallels making equal intercepts on a line. This diagram is not common in geometry, but we often have a diagram containing three parallels cutting two other lines. We can make an important deduction about such a figure.

Suppose we have a figure (Fig. 22) in which three parallels cut one line in the points A, B and C and cut another line in the points X, Y and Z. Then if $\frac{AB}{BC} = \frac{m}{n}$, where m and n are whole numbers, we can divide AB into m equal parts and BC into n equal parts, and all these parts will be equal. If, further, we draw the parallels through all the points of division, these parallels will divide XY into m equal parts and YZ into n equal parts, and all these parts will be equal (to one another).

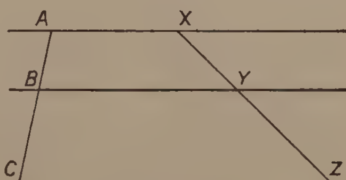


FIG. 22

Thus, $\frac{XY}{YZ} = \frac{m}{n} = \frac{AB}{BC}$. Indeed, we can say that $\frac{XY}{YZ} = \frac{AB}{BC}$ even though we do not know the actual values of m and n.

We thus have the result that if three parallels cut one line in A, B and C and another line in X, Y and Z, then $\frac{AB}{BC} = \frac{XY}{YZ}$ or, as a statement:

The ratio of the intercepts made by three parallel lines on one line is equal to the ratio of the intercepts which they make on any other line.

Note that the diagram which we have been using contains other equal ratios, for $\frac{AB}{AC} = \frac{XY}{XZ}$ and $\frac{BC}{AC} = \frac{YZ}{XZ}$ which can also be deduced by applying the Equal Intercept Theorem.

Note also that we have not proved this new theorem completely, for it is not always possible to express the ratio $\frac{AB}{BC}$ in the form $\frac{m}{n}$ where m and n are whole numbers.

The theorem, nevertheless, is true for all cases, and it may be referred to as the Proportional Intercept Theorem, since another way of stating it is:

The intercepts made by three parallel lines on a line are proportional to the intercepts made on any other line.

EXERCISE 6D

Apply the Proportional Intercept Theorem to the given diagrams.

Note: The answer to a question such as

"Find a ratio equal to $\frac{AN}{NB}$" should be a statement, or equation, with $\frac{AN}{NB}$ as subject.

Thus the answer to Question 1 is

$\frac{AN}{NB} = \frac{AM}{MC}$ (not just $\frac{AM}{MC}$).

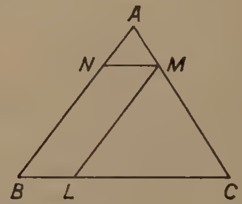


FIG. 23

1. In Fig. 23, $NM \parallel BC$ and $LM \parallel BA$. Find a ratio equal to $\frac{AN}{NB}$ and a ratio equal to $\frac{CL}{LB}$.

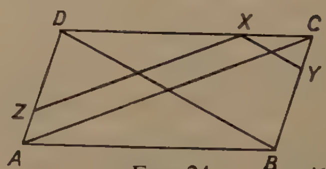


FIG. 24

2. In Fig. 23, prove that $\frac{AN}{NB} = \frac{BL}{LC}$.

3. In Fig. 23, prove that $BLMN$ is a parallelogram and use the results of 1 and 2 to prove that $\frac{AN}{NM} = \frac{ML}{LC}$.

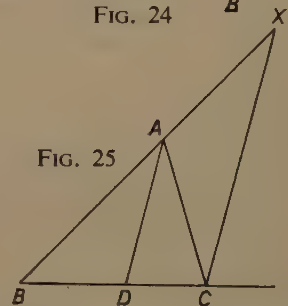


FIG. 25

4. In Fig. 24, $ABCD$ is a parallelogram and XY, XZ are parallel to BD, AC. Apply the Prop. Int. Theorem to find two ratios equal to $\frac{CX}{CD}$. Deduce from your answers that $AZ = CY$.

5. Use the result of 4 to prove that $AZCY$ is a parallelogram.

6. In Fig. 25, AD is parallel to CX and AD bisects the angle BAC. Apply the converse of the Isosceles Triangle Theorem to prove that $AX = AC$. By applying the Prop. Int. Theorem deduce that $\frac{BA}{AC} = \frac{BD}{DC}$.

7. In Fig. 23, given that $AN = 12$, $AM = 10$, $MN = 16$, and $NB = 42$, calculate MC and LC.

8. In Fig. 25, given that $AB = 8$, $AC = 6$ and $BC = 7$, calculate BD and DC.

9. *ABCD* is a parallelogram and *M*, *N* are the mid-points of *AB*, *CD*. Prove that the lines *DM* and *BN* divide the diagonal *AC* into three equal parts.

The Proportional Intercept Theorem Applied to a Triangle

Fig. 26 shows three parallel lines cut by two other lines. We can therefore apply the Prop. Int. Theorem to this diagram and deduce

$$\frac{AX}{XB} = \frac{AY}{YC} \text{ or } \frac{AX}{AB} = \frac{AY}{AC}.$$

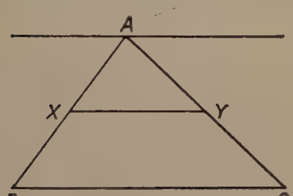


Fig. 26

Indeed, this is the most common use of the theorem, and this deduction forms a theorem of its own. The statement of this new theorem is:

A line parallel to a side of a triangle divides the other two sides in equal ratios.

This theorem may be applied to any diagram containing a triangle and the points where two of its sides are cut by a line parallel to the third side. Thus, in applying the theorem to Fig. 27, in which *MN* is parallel to *RS* we write:

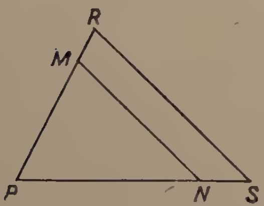


Fig. 27

$$MN \parallel RS \therefore \frac{PM}{MR} = \frac{PN}{NS}$$

$$\text{or } MN \parallel RS \therefore \frac{PM}{PR} = \frac{PN}{PS}$$

whichever is the more useful.

This theorem has a converse, which is often applied. The statement of this converse is:

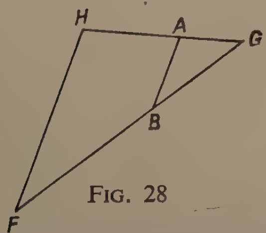


Fig. 28

A line which divides two sides of a triangle in equal ratios is parallel to the third side.

We apply this theorem to a diagram (Fig. 28) in which we know that two ratios are equal, and we deduce that two lines are parallel. Thus, in applying the theorem to Fig. 28, in which $\frac{AG}{AH} = \frac{GB}{BF}$ we write $\frac{AG}{AH} = \frac{GB}{BF}$

$$\therefore AB \parallel HF.$$

By means of these theorems we can deduce equal ratios from parallel lines and vice versa. They also provide proofs of the theorems about similar triangles, which we have accepted so far without proof. The theorem above is best proved by applying the Equal Intercept Theorem and ignoring the Proportional Intercept Theorem.

Given: G and H are points in AB and AC such that GH is parallel to BC (Fig. 29).

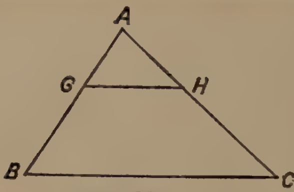


FIG. 29

To prove: $\dfrac{AG}{GB} = \dfrac{AH}{HC}$

Construction: Let $\dfrac{AG}{GB} = \dfrac{m}{n}$ where m and n are whole numbers.

Divide AG into m equal parts and GB into n equal parts, and through the points of division draw lines parallel to BC.

Proof: The parallel lines in the construction make equal intercepts on AB.

$\therefore$ They make equal intercepts on AC.

Now AH contains m of these intercepts and HC contains n.

$\therefore \dfrac{AH}{HC} = \dfrac{m}{n} \quad \therefore \dfrac{AG}{GB} = \dfrac{AH}{HC}$

Note that this proof applies only when the ratio can be expressed in the form $\dfrac{m}{n}$ where m and n are whole numbers. Many ratios can not be expressed in this form (e.g. the ratio of the diagonal of a square to the side of the square) but the proof for such ratios is too difficult to give here. We shall assume that the theorem is true for all ratios.

In the proof of the converse theorem we apply the theorem itself. Each sentence of the proof should be studied and its meaning understood.

Given: M and N are points on AB and AC such that $\dfrac{AM}{MB} = \dfrac{AN}{NC}$ (Fig. 30).

To prove: MN is parallel to BC

Construction: Draw MX parallel to BC, meeting AC at X.

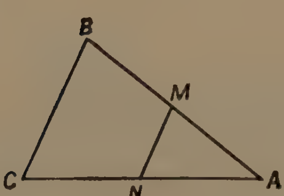


FIG. 30

Proof: $MX \parallel BC$ (construction)

$\therefore \dfrac{AM}{MB} = \dfrac{AX}{XC}$ (Theorem proved above)

But $\dfrac{AM}{MB} = \dfrac{AN}{NC}$ (given)

$\therefore \dfrac{AX}{XC} = \dfrac{AN}{NC} \quad \therefore \dfrac{AX}{AC} = \dfrac{AN}{AC}$

$\therefore AX = AN$

$\therefore$ N and X are the same point

$\therefore$ MN is parallel to BC.

Note that X and N are located by different facts. X is located by the fact that MX is parallel to BC; N is located by the fact that $\dfrac{AN}{NC} = \dfrac{AM}{MB}$.

The proof combines these two facts to deduce that they locate the same point. Thus, when we draw the parallel to *BC* through *M* we cannot prevent it from going through *N*. Thus we prove that *MN* is parallel to *BC*.

These theorems are used in proving the theorems about similarity, which we have so far applied without any proof. The proofs will be given later; in the meantime the theorems about similar triangles may be freely applied.

The First Equal Area Theorem

1. *Parallelograms on the same base and between the same parallels are equal in area.*

This theorem may be applied to a diagram containing two parallelograms which share a base, and have their opposite sides in the same straight line. The names of the two parallelograms should always be given, and the names will have two common letters. Thus, *ABPQ* and *ABXY*, if they are parallelograms, have a common base *AB*. To apply the theorem to them they require also to have *PQXY* in the same straight line. In applying this theorem to these two parallelograms, we should write:

ABPQ and *ABXY* are parallelograms.
PQ and *XY* are in the same straight line.
$\therefore$ $\|$ gm. *ABPQ* = $\|$ gm. *ABXY*.

EXERCISE 6E

1. In Fig. 31, *AB* is parallel to *DCFE*; *BE* is parallel to *AKFG*; *AD*, *BKC* and *GE* are parallel. Prove that *BKGE* = *ABCD*.
(Study these steps carefully:
Step 1: Apply a parallelogram theorem to prove that *ABCD*, *ABEF* and *BKGE* are parallelograms.
Step 2: Apply the equal area theorem above to prove that *ABCD* and *BKGE* are each equal to a third parallelogram in the figure.
Step 3: State the conclusion.)

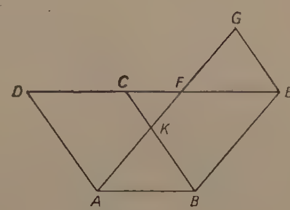


FIG. 31

2. In Fig. 32, *EF* is parallel to *ABCD*; *AE*, *GBKF* and *HD* are parallel; *AG*, *EKCH* and *FD* are parallel. Prove that *AEKG* = *DFKH*.

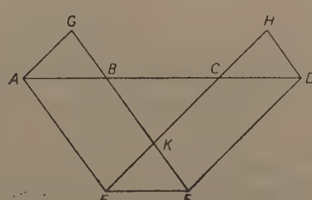


FIG. 32

3. In Fig. 33, *ABCD* and *GHLK* are parallelograms. *ABGH* are in a straight

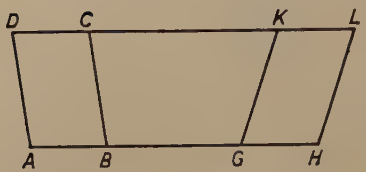


FIG. 33

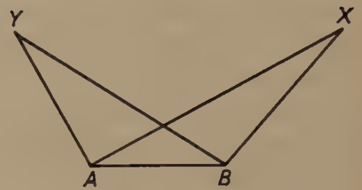


FIG. 34

line and $DCKL$ are in a straight line. Also $AB = GH$. Prove that $ABCD = GHKL$.

(*Step* 1: Prove that $ABKL$ is a parallelogram. *Step* 2: Apply the equal area theorem to deduce the result.)

4. ABC is a triangle. D is the mid-point of BC, E the mid-point of CA and F the mid-point of AB. Apply the mid-point theorem to prove that $AFDE$, $BDEF$ and $CEFD$ are parallelograms. Apply the equal area theorem to prove that these parallelograms are equal in area.

5. ABX and ABY are two triangles and AB is parallel to XY (Fig. 34). Prove that $\triangle ABX$ and $\triangle ABY$ are equal in area (or that $\triangle ABX = \triangle ABY$). (Two construction points are needed: K on XY such that $AK \parallel BX$ and L on XY such that $BL \parallel AY$. Write this under the heading "Construction." Apply the equal area theorem to prove that two parallelograms are equal in area and from this deduce that the two required triangles are equal in area.)

The Second Equal Area Theorem

The result in Question 5 is of great importance, and can be stated as our second area theorem:

If two triangles have the same base and the line joining their vertices is parallel to this base, then the triangles are equal in area.

To apply this theorem to a figure containing a line, PQ, parallel to a line AB, we write:

$$PQ \parallel AB \therefore \triangle ABP = \triangle ABQ.$$

No more than this is needed, but it must be observed that the same four letters appear in both the given fact and in the deduced fact.

Note also that from the given fact $PQ \parallel AB$, we could also deduce that $\triangle APQ = \triangle BPQ$, for these two triangles have the same base, PQ, and the line AB, joining their vertices, is parallel to this base.

We have seen earlier (Chapter 3) that the area of a triangle is measured by multiplying half the product of its base and its altitude (or height), and there are many sets of data from which we can deduce that two triangles

are equal in area. It would be possible to prove a large number of theorems in which the deduction, in each case, is that two triangles are equal in area, but it would not be worth while to do so.

It should be borne in mind that any statement (using the letters of the diagram) that two triangles have equal bases and equal altitudes may be used in order to deduce that the triangles have equal areas.

Here are some examples of such deductions:

In Fig. 35 the given facts are that $AB = CD$ and $ABCD$ is a straight line. We draw PM perpendicular to AD. Our deduction reads:

$AB = CD$. PM is the altitude of each triangle.

$\therefore \triangle ABP = \triangle CDP$.

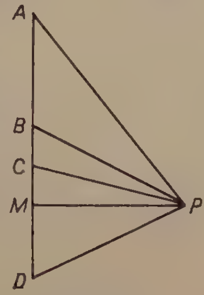


FIG. 35

In Fig. 36 the given facts are that $ABCD$ is a parallelogram, that X lies in CD and Y lies in AB. We write:

Draw XM, YN, altitudes.

Base AB = base CD ($ABCD$ is a $\|$ gm.).

Alt. XM = Alt. YN. ($XMYN$ $\|$ gm.).

$\therefore \triangle ABX = \triangle CDY$.

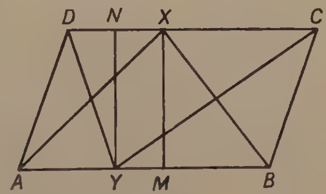


FIG. 36

In Fig 37 the given facts are that $ABCD$ is a straight line, that $AB = CD$ and that PQ is parallel to AD. We draw PM and QN perpendicular to AD and prove that $PM = QN$.

Base AB = base CD.

Alt. PM = alt. QN.

$\therefore \triangle ABP = \triangle CDQ$.

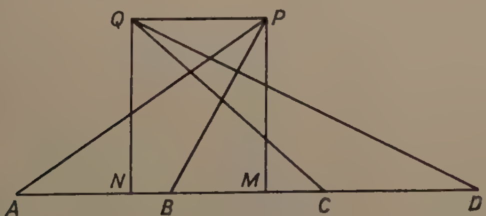


FIG. 37

In Fig. 38 the given facts are that M is the mid-point of the altitude AD and that $CE = BC$.

We write:

Base $BE = 2$ base BC.

Alt. $MD = \frac{1}{2}$ alt. AD.

$$\begin{aligned} \therefore \triangle ABC &= \tfrac{1}{2}BC.AD \\ &= BC.MD \\ &= \tfrac{1}{2}.BE.MD \\ &= \triangle BEM. \end{aligned}$$

$\therefore \triangle ABC = \triangle BEM$.

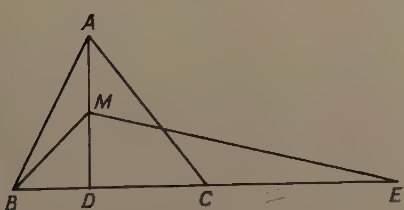


FIG. 38

In Fig. 39 C is the mid-point of XY. We shall deduce that $\triangle ABX = \triangle ABY$.

Construction: Draw XM and YN perpendicular to AB.

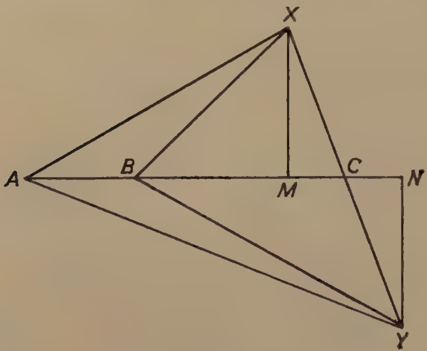


FIG. 39

In the triangles XMC and YNC,

$$XC = YC$$
$$X\hat{C}M = Y\hat{C}N \text{ (vert. opp.)}$$
$$M\hat{X}C = N\hat{Y}C \ (XM \parallel YN,$$

both perpendicular to AB)

$$\therefore \triangle XMC = \triangle YNC$$
$$\therefore XM = YN$$

Base $AB = AB$;

alt. $XM =$ alt. YN

$$\therefore \triangle ABX = \triangle ABY.$$

Note that Fig. 39 gives two equal triangles, ABX and ABY by making AB pass through the mid-point of XY. X and Y are any two points: C is the mid-point of XY and AB is any line passing through C.

Proof of the Equal Parallelogram Theorem

Given: $ABPQ$ and $ABXY$ are parallelograms and $PQXY$ are in a straight line.

To prove: $ABPQ = ABXY$.

Proof: In the triangles AQY and BPX

$$AY = BX \ (ABXY \parallel \text{gm.})$$
$$A\hat{Y}Q = B\hat{X}P \ (AY \parallel BX)$$

and since $A\hat{Q}Y = B\hat{P}X \ (AQ \parallel BP)$,

$$Q\hat{A}Y = P\hat{B}X$$
$$\therefore \triangle AQY \equiv \triangle BXP$$
$$\therefore \triangle AQY = \triangle BXP$$
$$\therefore ABPY - \triangle AQY$$
$$= ABPY - \triangle BXP,$$

i.e. $ABPQ = ABXY$.

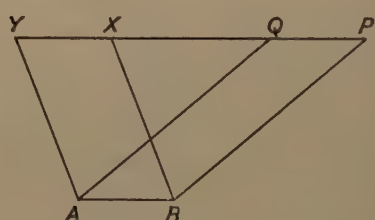


FIG. 40

Note the difference between $\triangle AQY \equiv \triangle BXP$ and

$$\triangle AQY = \triangle BXP.$$

The former states that the triangles agree about all their parts (and therefore about their areas). The second states that the triangles have equal areas.

From $\triangle AQY \equiv \triangle BXP$ we can deduce $\triangle AQY = \triangle BXP$, but the converse of this cannot be used.

CHAPTER 7

PYTHAGORAS' THEOREM

THIS THEOREM is one of the most important in geometry, and also one of the best-known. It was known, and applied, long before Pythagoras first proved it in the sixth century B.C., but it has since then always been associated with his name.

Statement of the Theorem. If a triangle is right-angled, then the square on its hypotenuse is equal to the sum of the squares on the other two sides.

The given fact is that a certain angle is a right-angle. The deduced fact is that the square on the hypotenuse is equal to the sum of the squares on the other two sides.

Application of the Theorem. In applying the theorem we state the name of the right-angle and deduce the result about the squares on three lines. The given or the deduced fact may be placed first and the other second, but they should always both be written. Thus:

$A\hat{B}C$ is a right-angle, $\therefore$ $AC^2 = AB^2 + BC^2$. Or,

$DE^2 + DF^2 = EF^2$ ($E\hat{D}F$ is a right-angle).

In the first of these the given fact is placed first, and the deduced fact follows the sign $\therefore$ In the second, the deduced fact is placed first and the given fact is placed in brackets to indicate that it is the reason for the deduced fact.

Note that the given and deduced facts both refer to the same three letters, and this rule should always be observed. It is sometimes permissible to write "By Pythagoras, $AC^2 = AB^2 + BC^2$," but only if ABC is the only right-angle in the diagram; or "$DE^2 + DF^2 = EF^2$ (Pyth.)," but only if $E\hat{D}F$ is the only right-angle in the diagram. The best and safest way to apply the theorem is to give the name of the right-angle in every case.

The expression AB^2 has two meanings. First, it can mean the area of a square whose side is equal to AB. With this meaning, the theorem states that the sum of two *areas* is equal to a third *area*, but it is never necessary to draw the three squares in the diagram. Secondly, AB^2 can mean the *number* obtained by squaring the *number* of units (of length) in AB.

With this second meaning the theorem states that the number obtained by squaring one number is equal to the sum of the numbers obtained by squaring two others, and this is the meaning we adopt when we apply the theorem to calculation problems.

The Converse of Pythagoras' Theorem

Statement. If the square on one side of a triangle is equal to the sum of the squares on the other two sides, then these two sides form a right-angle.

This is one of many ways of stating this converse, and any way of stating it is acceptable which indicates that the given fact is the fact about the squares, and the deduced fact is the right-angle fact. But the word "hypotenuse" must on no account be used in stating this (converse) theorem, for there is no known right-angled triangle and so there can be no hypotenuse.

Application of the Theorem. The converse is applied in the same way as Pythagoras' theorem, except that the given and the deduced facts are interchanged. Thus:

$AC^2 = AB^2 + BC^2$. $\therefore$ $A\hat{B}C$ is a right-angle.

$E\hat{D}F = 90°$ ($DE^2 + DF^2 = EF^2$).

Note that the given and deduced facts again refer to the same three letters. An application of either theorem which ignores this rule may be faulty.

In each of the two examples which follow, we give a sound and an unsound method of obtaining the result.

Example 1: In the triangle ABC, $AB = 12$, $AC = 5$ and $\hat{A} = 90°$. Calculate BC.

Sound Method: $BC^2 = AB^2 + AC^2$ ($B\hat{A}C$ is a right-angle)

$= 144 + 25$

$= 169$

$\therefore$ $BC = 13$.

Unsound Method: $BC^2 = 144 + 25$ (Pyth.)

$= 169$, etc.

(In the unsound method the application of the theorem is not made clear.)

Example 2: The sides of a triangle are 8, 15 and 17. Prove that the triangle is right-angled.

Sound Method: Let $AB = 8$, $BC = 15$ and $CA = 17$

Then $AB^2 + BC^2 = 64 + 225 = 289$

and $AC^2 = 289$

$\therefore$ $AB^2 + BC^2 = AC^2$

$\therefore$ $A\hat{B}C$ is a right-angle, $\therefore$ The triangle is right-angled.

Unsound Method: $8^2 + 15^2 = 17^2$

$\therefore$ $289 = 289$

$\therefore$ The triangle is right-angled.

This may seem to answer the question, but it is quite worthless. The right-angle is not deduced from the arithmetical fact that $8^2 + 15^2 = 17^2$, but from the geometrical fact that the square on one side of a triangle is equal to the sum of the squares on the other two sides. A geometrical fact should never be deduced from an arithmetical fact.

EXERCISE 7A

In the following questions Pythagoras' theorem should be applied to each right-angled triangle in the diagram. If no letters are given in the question (as in No. 7, 9 and 10) they should be supplied, as we have just shown in the worked Example 2. Right-angles are indicated in the given diagrams by squares.

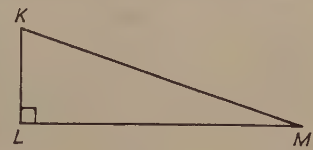


FIG. 1

1. In Fig. 1, prove that $LM^2 = KM^2 - KL^2$.

2. In Fig. 2, prove that $XY^2 - XW^2 = WZ^2 - YZ^2$.

3. In Fig. 3, prove that $QP^2 - QR^2 = SP^2 - RS^2$.

4. In Fig. 4, $AB = 5$ and $AC = 12$. Calculate BC, giving the answer (i) exactly, as a root and (ii) approximately, correct to 2 decimal places.

5. In Fig. 5, $OA = 17$, $OB = 10$ and $OM = 8$. Calculate AB and the area of the triangle OAB.

6. $ABCD$ is a square. Prove that $AC^2 = 2AB^2$.

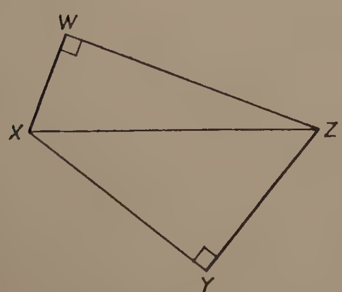


FIG. 2

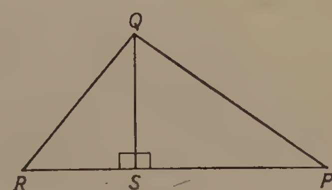


FIG. 3

FIG. 4

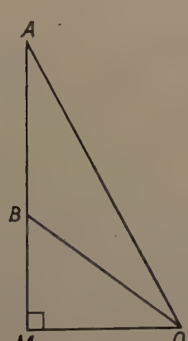


FIG. 5

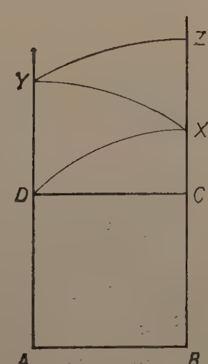


FIG. 6

7. Calculate the length of the diagonal of a square of side 12 inches, giving the answer (i) exactly and (ii) correct to 2 places of decimals.

8. In Fig. 6, the sides AD and BC of a square $ABCD$ are produced as shown. X, Y and Z are located from the facts $BX = BD$, $AY = AX$, $BZ = BY$. By applying Pythagoras' theorem to the triangles ABD, ABX, ABY and ABZ, prove that $CZ = AB$. (It is not necessary to work out any square roots for this question.)

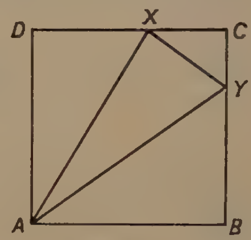


FIG. 7

9. One side of a rectangle is 7 cm long and each diagonal is 11 cm long. Calculate the area of the rectangle, giving the answer correct to 2 places of decimals.

10. The sides of a right-angled triangle are x, $x - 1$ and $x - 8$ cm, respectively. Find the value of x.

In the following questions the converse of Pythagoras' theorem should be applied in proving angles to be right-angles. Pythagoras' theorem may, of course, be applied to any triangle containing a *known* right-angle.

11. $PQ = 24$, $QR = 7$ and $PR = 25$. Prove that PQR is a right-angle.

12. $KL = 2$, $KM = 3$ and $LM = \sqrt{13}$. Prove that LKM is a right-angle.

13. In Fig. 7, $ABCD$ is a square, $AB = 9$, $DX = 6$ and $BY = 7$. Prove that AXY is a right-angle.

14. Prove that a triangle whose sides are $x^2 + 4$, $x^2 - 4$ and $4x$, respectively, is right-angled.

15. In Fig. 8, $AB = 7$, $BC = CD = 6$ and $DA = 11$. ABC is a right-angle. Prove that ACD is also a right-angle.

FIG. 8

16. Prove that the triangle whose sides are $m^2 + n^2$, $m^2 - n^2$ and $2mn$ is right-angled.

(If whole numbers are substituted for m and n in these three expressions, three numbers will be obtained which will give the lengths of the sides of a right-angled triangle.

E.g. $m = 2$ and $n = 1$ give 5, 3 and 4, a well-known right-angled triangle. $m = 4$, $n = 3$ give the sides of the triangle in No. 11, above. It can be shown that all right-angled triangles whose sides are each an exact number of units can be obtained from these expressions.)

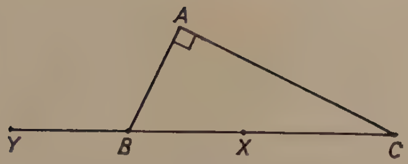


FIG. 9

17. $ABCD$ is a quadrilateral in which A is a right-angle, $AB = 8$, $BC = 7$, $CD = 4$ and $DA = 1$. Prove that C is also a right-angle.

18. In Fig. 9, BAC is a right-angle, and $BX = BA = BY$. Prove that $CA^2 = CX.CY$.

Co-ordinates

Fig. 10 shows a square of side 10 units, divided into 100 unit squares. Each corner of each unit square may be located by giving the number of horizontal and vertical unit steps required to reach it from O. Thus, to reach the point marked A we take 4 horizontal steps from O and then 7 vertical steps.

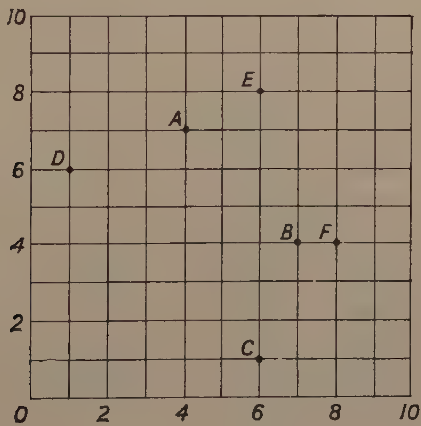


FIG. 10

These two numbers, 4 and 7, are called the *co-ordinates* of A and we say that A is the point (4,7). Before A can be located by these two numbers, we must know where O is, which direction is taken as horizontal, and the size of the unit. We also agree that the horizontal step is given first. Thus, B is the point (7,4). This is also written $B \equiv (7,4)$.

If, starting at the point D, (1,6), we travel 3 units to the right and then 1 unit upwards we arrive at the point A. The first of these is called the *run* from D to A, and the second is called the *rise* from D to A. The run is measured to the right, and the rise is measured upwards. Thus, the run from D to B is 6 and the rise is -2. The negative sign indicates that we go two steps downwards.

The run from B to C is -1 and the rise is -3. When the co-ordinates of two points are given the run and rise from one to the other are easily found by subtraction. Thus the run from C (6,1) to B (7,4) is $7 - 6 = 1$, and the rise is $4 - 1 = 3$. The run from A (4,7) to C (6,1) is $6 - 4 = 2$ and the rise is $1 - 7 = -6$.

To calculate the distance between two points whose co-ordinates are given we apply Pythagoras' theorem to the triangle whose hypotenuse is the required distance, and whose other sides are the run and the rise between the two points. Thus:

$$BC^2 = 1^2 + 3^2 = 10, \therefore BC = \sqrt{10}.$$
$$DB^2 = 6^2 + (-2)^2 = 36 + 4 = 40, \therefore DB = \sqrt{40}.$$

Note that even though the rise is negative in this case, its square is positive.

EXERCISE 7B

The following questions refer to Fig. 10.

1. Give the co-ordinates of E and F.

2. Calculate the run and the rise from D to E and from D to F.

3. Calculate the values of AE^2, EF^2 and AF^2.

4. Deduce from the results of Question 3 that AEF is a right-angle.

5. Prove that the angle DAC is a right-angle.

6. Calculate the run, the rise and the distance from G to H where $G \equiv (12,5)$ and $H \equiv (17,17)$.

7. Calculate the run, the rise and the distance from K to D where $K \equiv (-2,2)$ (i.e. the run from O to K is -2, and the rise from O to K is 2).

8. If $P \equiv (x,y)$ write an expression for AP^2.

9. Prove that the triangle OGH is isosceles (G and H are the points in Question 6).

The run, rise and distance between two points can be calculated even when the points are not at the corners of squares. Thus, the run from (2·7,3·8) to (4·8,5·8) is
$4{\cdot}8 - 2{\cdot}7 = 2{\cdot}1$,
the rise is $5{\cdot}8 - 3{\cdot}8 = 2{\cdot}0$
and the distance is $\sqrt{(2{\cdot}1^2 + 2{\cdot}0^2)} = \sqrt{8{\cdot}41} = 2{\cdot}9$.

10. Calculate the run, rise and distance between the following pairs of points.
(a) (3·7,2·6) and (4·5,4·1)
(b) (1·2,5·7) and (4·7,6·9)
(c) $(2\frac{1}{2},6\frac{1}{4})$ and $(4\frac{3}{4},9\frac{3}{4})$.

Proof of Pythagoras' Theorem

There are many proofs of this famous theorem (first stated on page 318) and one of the simplest is that which applies the properties of similar triangles.

Given: $B\hat{A}C$ is a right-angle.

To prove: $BC^2 = AB^2 + AC^2$.

Construction: Let AD be the perpendicular from A to BC (Fig 11).

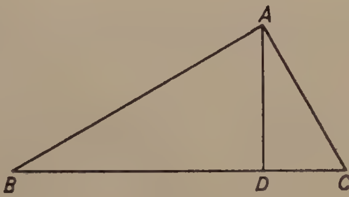


FIG. 11

Proof: In the $\triangle$s ABD and CBA

$$\hat{B} = \hat{B}$$

$$A\hat{D}B = C\hat{A}B \text{ (right-angles)}$$

$$\therefore \triangle ABD \parallel\!| \triangle CBA$$

$$\therefore \frac{AB}{BD} = \frac{CB}{BA}$$

$$\therefore AB^2 = BD.BC$$

$$\text{Similarly } AC^2 = CD.BC^*$$

$$\therefore AB^2 + AC^2 = BD.BC + CDBC$$

$$= BC(BD + CD)$$

$$= BC.BC = BC^2$$

$$\therefore BC^2 = AB^2 + AC^2.$$

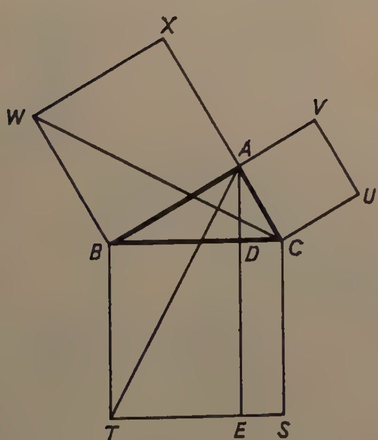


FIG. 12

The proof just given treats BC^2 as the number obtained by squaring the number of units in BC. We give now the proof originally given by Euclid, although we shall use the symbol BC^2 to represent what Euclid called, in full, the area of the square on BC.

Given: BAC is a right-angle.

To prove: $BC^2 = AB^2 + AC^2$.

Construction: Let $ABWX$, $ACUV$ and $BCST$ be the squares on AB, AC and BC respectively. Let the line through A perpendicular to BC meet BC at D and ST at E (Fig. 12).

* Note that this statement is the same as that in the preceding line but with B and C interchanged. The given fact is unchanged when we interchange B and C and so any deduced fact obtained by interchanging B and C will also be true. The word *similarly* should be reserved for this particular kind of interchange of letters.

The object of this proof is to show that the square ABWX and the rectangle BDET are equal in area. A similar proof will show that the square ACUV and the rectangle CDES are equal in area. It will then follow that the square BCTS is equal in area to ABWX and ACUV together.

Step 1. *Prove* $\triangle ABT = \triangle WBC$ (*in area*).

In the $\triangle$s ABT, WBC

$AB = WB$ (sides of a square)

$BT = BC$ (sides of a square)

$A\hat{B}T = W\hat{B}C$ (rt.-angle $+ A\hat{B}C$)

$\therefore \triangle ABT \equiv \triangle WBC$

$\therefore \triangle ABT = \triangle WBC.$

Step 2. *Prove* $\triangle WBC = \frac{1}{2}ABWX$.

Now BAX is a right-angle (in a square) and BAC is a right-angle (given).

$\therefore$ CAX is a straight line. *This simple step should never be omitted. It is the only use we make of the given fact.*

$\therefore \triangle WBC = \frac{1}{2}ABWX$, being on the same base (WB) and between the same parallels.

Also $\triangle ABT = \frac{1}{2}BDET$, being on the same base (BT) and between the same parallels.

$\therefore ABWX = BDET$ since $\triangle ABT = \triangle WBC$

Similarly $ACUV = CDES$

$$\begin{aligned} \therefore AB^2 + AC^2 &= ABWX + ACUV \\ &= BDET + CDES \\ &= BCST \\ &= BC^2 \end{aligned}$$

$$\therefore BC^2 = AB^2 + AC^2.$$

Pythagoras' theorem may also be illustrated in the accompanying diagram (Fig. 13) in which the shaded triangles 1, 2 and 3 in the upper figure are moved to new positions as shown in the lower figure.

The area left unshaded in each figure is the same.

In the upper figure the unshaded area consists of squares with sides equal to AB and AC, and in the lower figure the unshaded area is the square with the side BC. But this does not constitute a "proof" of the theorem.

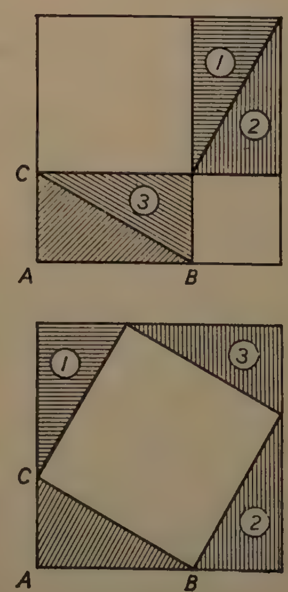


FIG. 13

Proof of the Converse of Pythagoras' Theorem

Given: $AB^2 + AC^2 = BC^2$.
To prove: BAC is a right-angle.
Construction: Let X, Y and Z be three points such that $XY = AB$, $XZ = AC$ and $Y\hat{X}Z$ is a right-angle (Fig. 14).

X may be plotted anywhere. Y is partly located by the fact that $XY = AB$ and Z is then completely located by the other two facts. There is nothing to prevent X being plotted at A and Y at B but it is more convenient to plot the new points so that the two triangles do not overlap. It should be noted how the given and construction facts are used in the proof.

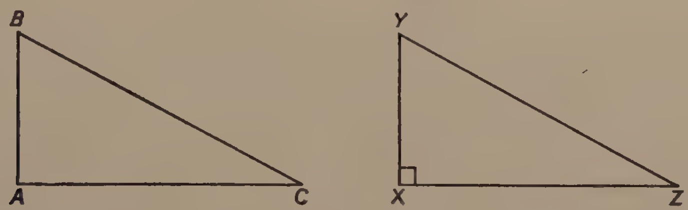


FIG. 14

Proof:	$Y\hat{X}Z$ is a right-angle	*Construction Fact* 3.
	$\therefore\ YZ^2 = XY^2 + XZ^2$	
	$= AB^2 + AC^2$	*Construction Facts* 1 *and* 2.
	$= BC^2$	*Given Fact.*
	$\therefore\ YZ = BC$.	
	In the triangles ABC and XYZ	
	$AB = XY$	*Construction Fact* 1.
	$AC = XZ$	*Construction Fact* 2.
	$BC = YZ$ (proved)	
	$\therefore\ \triangle ABC \equiv \triangle XYZ$	
	$\therefore\ B\hat{A}C = Y\hat{X}Z$	
	$=$ a right-angle.	*Construction Fact* 3.
	$\therefore\ BAC$ is a right-angle.	

As before, the *italics* are not part of the proof; they are intended to explain the ideas behind the proof.

CHAPTER 8

CIRCLES

A CIRCLE consists of all the points in a plane whose distances from a given point (in the plane) are equal. The given point is called the centre (of the circle). The distance of the centre (of a circle) from every point (of the circle) is called *the radius* (of the circle). A line starting at the centre of a circle and ending at a point of the circle is called *a radius* (of the circle). Note that *the radius* is a single length (such as 4 inches or 2 miles), and *a radius* is one of many lines which start at the centre and end at a point on the circle. In Fig. 1, C is the centre of the circle, CK is *a radius* (of the circle) and the length of CK is *the radius* (of the circle).

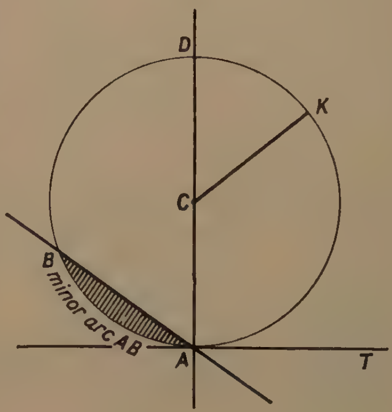


FIG. 1

Through a point A of the circle many lines can be drawn. Each of these lines may be regarded as "going into" the circle at A and as "coming out" at another point (of the circle). Such a line is called a *secant* of the circle (because it *cuts* the circle). Of the secants which can be drawn through A there are two which deserve special mention. The first is the secant which passes through the centre. Such a secant is called *a diameter* (of the circle); in this case it would be called *the diameter through A*.

The diameter of a circle is, like the radius, a length, and it is actually twice the radius. The second of the two special secants is the one which "comes out" at the same point as it "goes in." Such a secant is called *a tangent* to the circle and, in this case, it would be called *the tangent* (*to the circle*) *at A*. A tangent to a circle is thus a secant which goes in and comes out at the same point.

That part of a secant of a circle which lies inside the circle is called a *chord* (of the circle) and we see that of all the secants of a circle a diameter is one which has the longest chord and a tangent is one which has the shortest, of zero length.

A chord of a circle divides the circle into two areas, each of which is bounded by the chord and a part of the circle. Such a figure is called *a segment* (*of the circle*) and the curved part of its boundary is called *an arc* (*of the circle*). When the chord happens to be a diameter (of the circle) either segment is called *a semicircle*. When the chord is not a diameter, then the larger segment and the longer arc are called the *major segment* and *major arc*; the smaller segment and arc are called the *minor segment* and the *minor arc*.

A segment of a circle is bounded by a chord and an arc. If we now join the two ends of the chord to a point on the arc, these two lines will form an angle. This angle is called *an angle in the segment*. Many *angles* can be drawn *in a given segment* and we shall see later that all the *angles in a given segment* have the same size, so that we can talk of *the* angle in a given segment when we are concerned only with its size.

When the segment is a semicircle, the chord is a diameter and the angle will be called *an angle in the semicircle*. This phrase, then, refers to the angle formed by joining the ends of a diameter (of a circle) to a point on the circle. We shall see later that the size of all such angles, in all circles, is a right-angle and we shall say that *the angle in a semicircle is a right-angle*. This means that, given *any* semicircle, any *angle in it* is a right-angle.

A chord of a circle divides the circle into major and minor segments and we shall see later that *the angles in the two segments* are supplementary, i.e. the sum of any *angle in the major segment* and any *angle in the minor segment* is 2 right-angles. It is most important clearly to understand the meanings of these phrases so that we can apply theorems which use such phrases.

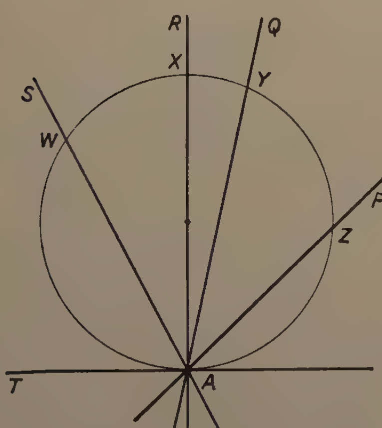


FIG. 2

EXERCISE 8A

In the following questions the answer to each should be a grammatical sentence, using, as far as possible, the words of the question, e.g. the first answer should be, *AP* comes out of the circle at *Z*.

1. Fig. 2 shows secants "going into" the circle at *A*. Name the points where *AP*, *AQ*, *AR*, *AS*, *AT* "come out of" the circle.

2. In Fig. 2, name:

(a) a point on the minor arc *AY*;

(b) a point on the major arc *AZ*;

(c) an angle in the segment bounded by chord AY and arc AZY;

(d) an angle in the segment bounded by chord AY and arc $AWXY$.

3. A theorem states that the angle between a tangent (AT) and a chord (AW) is equal to the angle in the segment on the other side of the chord (AW) from the tangent (AT). Apply this theorem to Fig. 2.

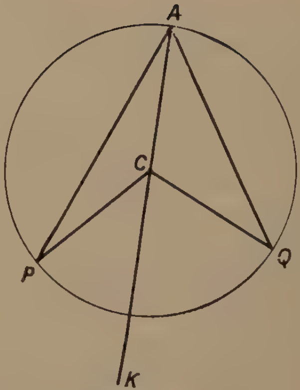


FIG. 3

4. A theorem states that angles in the same segment are equal.

Give three equal angles obtained by applying this theorem to the major segment of AW in Fig. 2.

5. A theorem states that the angle in a semicircle is a right-angle. Apply this theorem to Fig. 2 to obtain three right-angles.

(AX is a diameter of the circle.)

6. In Fig. 3, C is the centre of the circle. Name three equal lines. Name two isosceles triangles, giving the names of the pairs of equal sides. Name two pairs of equal angles from these isosceles triangles.

7. In Fig. 3, given that $P\hat{A}K = 23°$, calculate $P\hat{C}K$. Given also that $Q\hat{A}K = 32°$, calculate $P\hat{C}Q$.

8. In Fig. 4, $X\hat{A}Y$ may be described as an angle in the major segment XY of the circle centre M. How could the angles XBY and XCY be described?

9. In Fig. 4, XA and XB are tangents. State which circles they are tangents to. Give the names of the points in which XA enters and leaves the circle centre M; also the circle centre N. Repeat for XB.

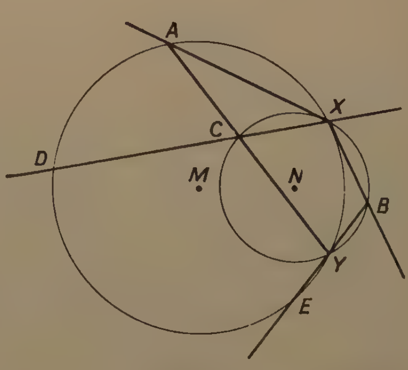


FIG. 4

10. In Fig. 4, a third line enters the circles at X. Give the names of the points where it leaves the two circles.

Two lines also enter the circles at Y. Give the names of the points at which they leave.

(Now compare your answers with the model answers given on page 450.)

The Use of the Term "Subtend"

An "angle in a segment" is obtained by joining a point on the arc (of the segment) to the ends of the arc (of the segment). A special word is used in connection with an angle obtained in this way, i.e. by joining one point to two others. The angle is said to be *subtended by* the two points *at* the single point, and the two points are said to *subtend* the angle *at* the point. We shall also speak of the *angle subtended by the line AB* or *by the arc AB* at a point P.

By this we shall mean the angle obtained by joining P to the ends A and B of the line (or arc). Thus we shall speak of "the angle subtended by an arc (of a circle) at the centre (of the circle)" or "the angle which a chord subtends at a point" and it should be quite clear which is the angle in question. The word *subtend* is used in stating theorems about angles connected with circles and its meaning should be clearly understood.

EXERCISE 8B

1. Draw a circle with centre C, a diameter XY and mark a point Z on the circle.

Name the angles which would be described as:

(a) The angle subtended by XZ at the centre (of the circle).

(b) The angle which a diameter subtends at a point (of the circle).

(c) The angle subtended at the centre by a diameter.

(d) The angle at Y subtended by XZ.

Describe in words the angles YXZ, YCZ and XZC.

2. What is the size of the angle subtended by a diameter of any circle at the centre (of the circle)?

3. A theorem states that the angles subtended at the centre of a circle by two equal arcs are equal. Apply this theorem to a figure in which PQ and QR are equal arcs of a circle centre C.

4. AB is a diameter of a circle centre C and P is a point on the circle. Name the angle subtended by AP at the centre. Name the angle subtended by AP at B. Prove that the first of these two angles is twice the second.

5. In the figure of Question 4 draw a smaller circle with centre C, cutting CA at a, CB at b and CP at p.

Prove that ap is parallel to AP.

6. Using the result of Question 4, prove that if XY and YZ are equal arcs of a circle and XA, YB are diameters, then $X\hat{A}Y = Y\hat{B}Z$.

Theorem. The angle which an arc (of a circle) subtends at the centre (of the circle) is double the angle which the arc subtends at a point on the remaining arc of the circle.

This theorem may be applied to any diagram containing the centre of a circle and three points on the circle. If the centre is A and the three points P, Q and R, then the theorem applied to this diagram would give $P\hat{A}Q = 2P\hat{R}Q$ (also $P\hat{A}R = 2P\hat{Q}R$ and $Q\hat{A}R = 2Q\hat{P}R$).

If we are given two points (A and B, say) on a circle centre C, they divide the circle into a major and minor arc. The angle which the minor arc subtends at the centre is $A\hat{C}B$ and the angle which the major arc subtends at the centre is the reflex angle ACB. The phrase "angle ACB" (or "$A\hat{C}B$") should not be used for the reflex angle and the word *reflex* must be used when a reflex angle is intended.

Apply this theorem in the following exercise.

EXERCISE 8C

1. C is the centre of a circle, A is a point on the major arc PQ and B is a point on the minor arc PQ (Fig. 5).

Deduce that $P\hat{A}Q + P\hat{B}Q = 180°$.

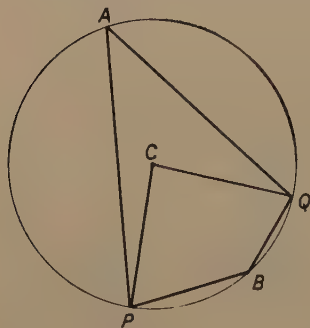


FIG. 5

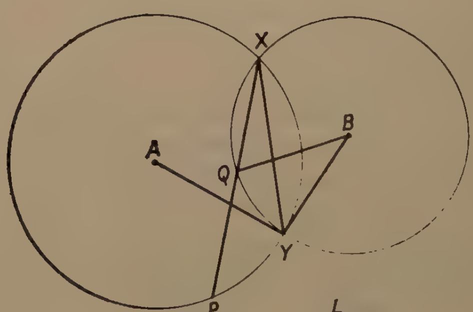


FIG. 6

2. AB is a diameter of a circle centre C. D is a point on the circle.

Deduce that $A\hat{D}B$ is a right-angle.

3. Apply the theorem to the arc PY and the point X of the circle centre A (Fig. 6). Apply it also to the arc QY and the point X of the circle centre B.

Deduce that $P\hat{A}Y = Q\hat{B}Y$.

4. Apply the theorem to the arc KU and the point E of the circle centre A; also to the arc LV and the point E of the circle centre B (Fig. 7). Deduce that $K\hat{A}U = L\hat{B}V$.

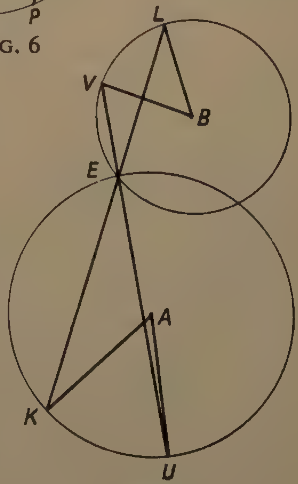


FIG. 7

5. In Fig. 8 apply the theorem to the arc PQ and the point A of the circle centre C; also to the point C and the arc PQ of the circle centre B. Deduce that $P\hat{B}Q = 4P\hat{A}Q$.

By applying the theorem to the arc PQ and the point E of the circle centre B, prove that $P\hat{E}Q = 2P\hat{A}Q$ and deduce that $E\hat{A}Q = E\hat{Q}A$.

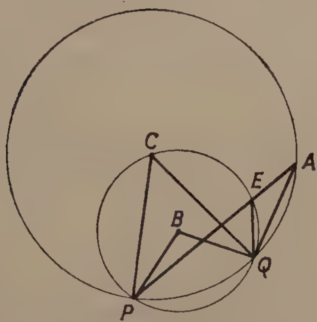


FIG. 8

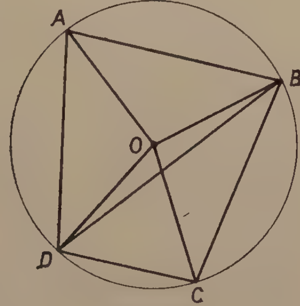


FIG. 9

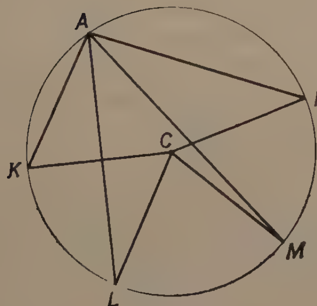


FIG. 10

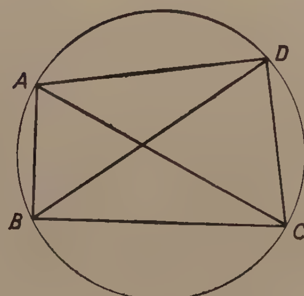


FIG. 11

6. In Fig. 9, the angle BOC is 100°, the angle COD is 60° and the angles DOA and AOB are equal (to one another). Calculate the angles of the triangles ABD and CBD. (Apply the theorem in calculating each of the six angles.)

7. In Fig. 10, KL and MN are two equal arcs. By applying the theorem prove that $K\hat{A}L = M\hat{A}N$.

Theorem. Angles in the same segment of a circle are equal.

We have seen that an angle in a segment of a circle is obtained by joining a point on the arc (of the segment) to the ends of the arc (of the segment). If several points on the arc are joined to the ends of the arc in this way we obtain several "angles in the same segment," and we can apply the theorem to such a diagram by saying that these angles are equal.

In Fig. 11 (page 395), C and D are the ends of an arc, and A and B are points on the arc. We apply the theorem to this figure by saying:

$$C\hat{A}D = C\hat{B}D \text{ (same segment).}$$

A and D are also the ends of an arc, and C and D are points on this arc, so we could apply the theorem again by saying

$$A\hat{B}D = A\hat{C}D \text{ (same segment).}$$

We could not, however, apply the theorem to the angles ABC and ADC, for these angles are not in "the same segment." There is another theorem (given later) which can be applied to these two angles.

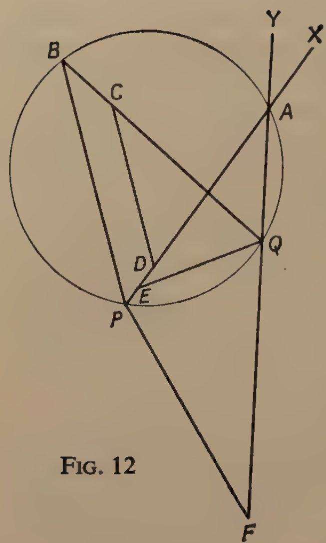


FIG. 12

EXERCISE 8D

Fig. 12 shows a circle with four points A, B, P and Q lying on it. It also contains other points which are described in the questions. You should draw a separate diagram for each question, showing the circle, A, B, P and Q and any other points described in the question.

1. Apply the theorem to segment $PQAB$.

2. PA is produced to X and QA to Y. Prove that $X\hat{A}Y = P\hat{B}Q$.

3. $QA = QE$. Prove that $A\hat{E}Q = P\hat{B}Q$.

4. $CD \parallel BP$. Prove that $D\hat{C}Q = D\hat{A}Q$.

5. $PF = PA$. Prove that $P\hat{B}Q = P\hat{F}Q$.

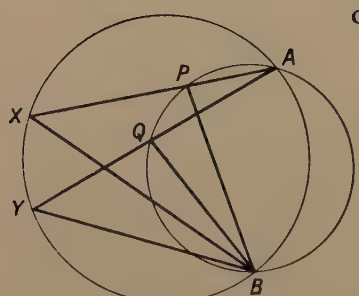


FIG. 13

6. Fig. 13 shows two circles intersecting at A and B, and two straight lines through A meeting the circles at P, X and Q, Y. Prove that $P\hat{B}Q = X\hat{B}Y$.

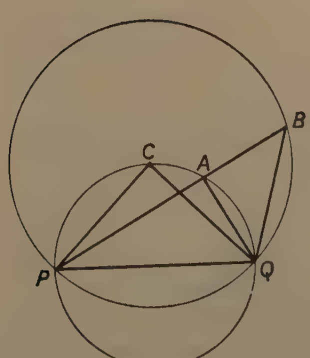


FIG. 14

7. Fig. 14 shows a circle, centre C, and another circle through C, meeting the former at P and Q. A line through P meets the first circle at A and the second circle at B.

Prove that $P\hat{A}Q = 2P\hat{B}Q$. Deduce that $BA = AQ$ by considering the angles of the triangle BAQ.

8. Fig. 15 shows a line XPQ going into a circle at P and coming out at Q; also a line XRS, going into the circle at R and coming out at S.

A is a point on the arc PQ.

Prove that $Q\hat{A}S - P\hat{A}R = P\hat{X}R$.

9. Two chords, KL and MN of a circle meet at a point A inside the circle. B is a point on the arc LM.

Prove that

$K\hat{B}M + L\hat{B}N = K\hat{A}M$.

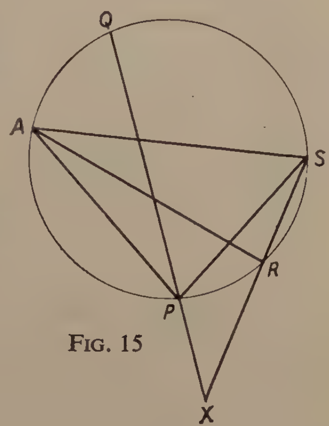


FIG. 15

Cyclic Quadrilaterals and Concyclic Points

If, in Fig. 11 (page 395), the circle is erased, we have a quadrilateral $ABCD$ with the property described on page 349, namely, a quadrilateral in which one side subtends equal angles at the ends of the opposite side; for CD subtends equal angles ($C\hat{A}D$ and $C\hat{B}D$) at the ends of the opposite side, AB.

We said then that such a quadrilateral was called a cyclic quadrilateral and the reason now appears. A cyclic polygon is one with the property that a circle can be drawn to pass through all its vertices. Thus, if we wish to draw a cyclic quadrilateral we draw a circle and take any four points on it as vertices of the quadrilateral. (It will still be a cyclic quadrilateral if the circle is erased.)

If we are given a quadrilateral and wish to find out whether it is cyclic, we draw the circle through three of its vertices. If this circle passes through the fourth vertex, the quadrilateral is cyclic, and if the circle does not pass through the fourth vertex, then the quadrilateral is not cyclic.

The word concyclic has a similar meaning, but it is used only about points. Thus four points are said to be concyclic if a circle can be drawn to pass through all four. If we are instructed to take four concyclic points, we draw a circle and mark four points on it. (They will still be concyclic if the rest of the circle is erased.) If we are given four points, we can draw the circle passing through three of them; if this circle also passes through the fourth, then the set of points is called concyclic. Cyclic, then, relates to polygons and concyclic relates to sets of points. We should never speak of a concyclic quadrilateral, or of four cyclic points.

If we are given a cyclic quadrilateral, we can immediately deduce the equal angle property by applying the theorem of "angles in the same

segment." The converse is also true, i.e. if we are given a quadrilateral with the equal angle property, we can immediately deduce that the quadrilateral is cyclic, and this theorem is the converse of the "same segment" theorem.

It can be applied freely to any quadrilateral with the property that one of its sides subtends equal angles at the ends of the opposite side.

In Fig. 16 $BA = BC$ and $DA = DE$.
We shall deduce that B, C, D and E are concyclic.

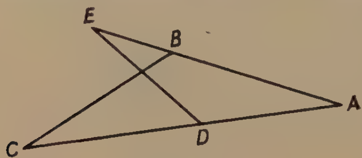


FIG. 16

$$D\hat{E}A = D\hat{A}E\ (DA = DE)$$
$$= C\hat{A}B$$
$$= A\hat{C}B\ (BA = BC).$$
$\therefore\ D\hat{E}B = D\hat{C}B$
$\therefore$ B, C, D and E are concyclic (conv. same seg.).

Note that in applying this theorem, the given fact ($D\hat{E}B = D\hat{C}B$) and the deduced fact (B, C, D, E concyclic) contain the same letters.

EXERCISE 8E

1. *FGHK* is a cyclic quadrilateral.

The diagonals FH and GK are produced to L and M such that LM is parallel to HK.

Prove that F, G, L and M are concyclic.

2. C is the centre of the circle PQR.

The bisector of the angle QCR meets PR at S (Fig. 17).

Prove that C, P, Q and S are concyclic.

3. $ABCD$ is a parallelogram and AD is produced to E such that $AB = BE$ (Fig. 18).

Prove that B, C, D and E are concyclic.

4. A, B, C, D and E are concyclic and BD bisects the angle CBE. AC meets BD at X and AD meets BE at Y (Fig. 19).

Prove that A, B, X and Y are concyclic.

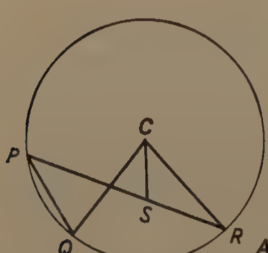


FIG. 17

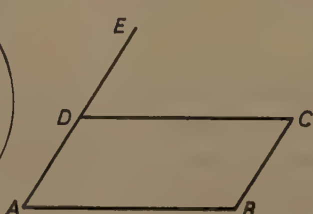


FIG. 18

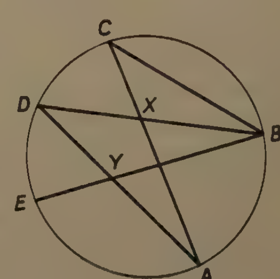


FIG. 19

Theorem. The opposite angles of a cyclic quadrilateral are supplementary.
Theorem. The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle (of the quadrilateral).

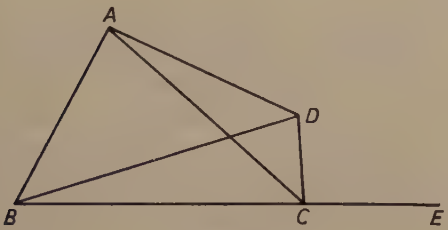


FIG. 20

These theorems may be applied to any diagram containing a cyclic quadrilateral as the given fact. The deduced fact may then be stated immediately. Fig. 20 shows a cyclic quadrilateral $ABCD$ with one side, BC, produced to E. (The circle containing A, B, C and D has been erased.) From this diagram we can deduce many facts about angles:

$B\hat{A}C = B\hat{D}C$ (same seg.) and three others by applying this theorem.
$B\hat{A}D + B\hat{C}D = 2$ rt.-angles, or 180°, by applying the first of the two theorems above (or $A\hat{B}C + A\hat{D}C = 180°$) and $D\hat{C}E = B\hat{A}D$ by applying the second of these two theorems.

An "exterior" angle of a quadrilateral is the angle formed by producing one of the sides. The "interior opposite" angle is the (interior) angle at the opposite vertex.

EXERCISE 8F

1. In Fig. 20, name the angles equal to $D\hat{A}C$ and $A\hat{B}D$ by applying the "same segment" theorem.

If CD is produced (beyond D) to F, name the angle equal to $A\hat{D}F$ by applying the "exterior angle" theorem.

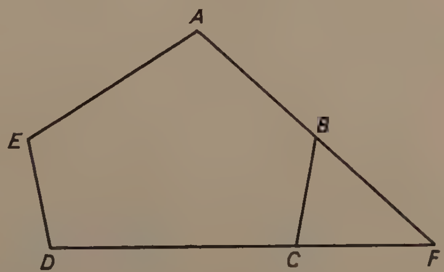


FIG 21

2. Fig. 21 shows five concyclic points, A, B, C, D and E. AB and DC meet at F. Other pairs of points may be joined as necessary. In each question say which theorem you apply to obtain the result. Specimen answer to (i):
$A\hat{E}C = C\hat{B}F$ (ext. angle cyc. quad.)
$A\hat{E}C = A\hat{D}C$ (same seg.)

(i) Name two angles equal to $A\hat{E}C$.

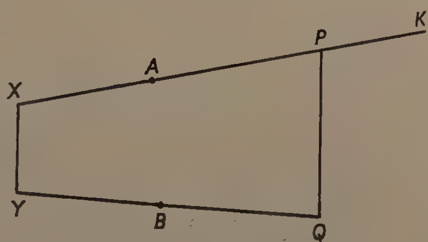


FIG. 22

(ii) Name an angle equal to $E\hat{B}C$.
(iii) Name two angles equal to $B\hat{C}F$.
(iv) Name an angle equal to $A\hat{E}D$.
(v) Name two angles supplementary to $A\hat{B}C$.
(vi) Name two angles equal to $A\hat{C}E$.

3. Fig. 22 shows four concyclic points, A, B, X and Y, also four concyclic points A, B, P and Q. YBQ and $XAPK$ are straight lines. By applying the "exterior angle" theorem twice, prove that XY is parallel to PQ.

Converse Theorems. Each of these two theorems has a converse, and these converse theorems may be applied to deduce concyclic points or cyclic quadrilaterals from equal or supplementary angles. Thus, from $A\hat{B}C$ and $A\hat{D}C$ are supplementary (or $A\hat{B}C + A\hat{D}C = 180°$), we deduce $ABCD$ is cyclic (or A, B, C and D are concyclic) and from $B\hat{A}E = B\hat{C}D$ (where DA is produced to E) we can also deduce that $ABCD$ is cyclic.
(Compare the following examples with those on page 398.)

FIG. 23

Example: In Fig. 23, $BA = BC$ and $DA = DE$. We shall deduce that B, C, D and E are concyclic.
(The proof is a repetition of that given on page 398 (Fig. 16), except that instead of "conv. same seg.," we have conv. ext. angle cyclic quad.).

EXERCISE 8G

1. $FGHK$ is a cyclic quadrilateral. FG and KH are produced to L and M such that LM is parallel to GH. Prove that F, L, M and K are concyclic.

2. C is the centre of the circle PQR. The bisector of the angle QCR is produced to meet PR at S (Fig. 24).

Prove that C, P, Q and S are concyclic.

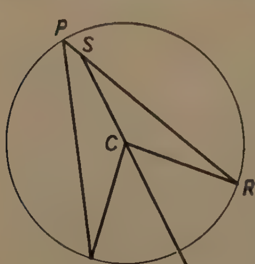


FIG. 24

3. $ABCD$ is a parallelogram and E is the point in AD such that $AB = BE$.

Prove that B, C, D and E are concyclic (Fig. 25).

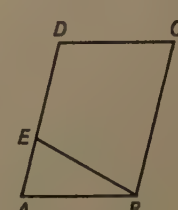


FIG. 25

4. A, B, C, D and E are concyclic and BD bisects the angle CBE. AC meets BD at X and DA produced meets BE at Y (Fig. 26).

Prove that A, B, X and Y are concyclic.

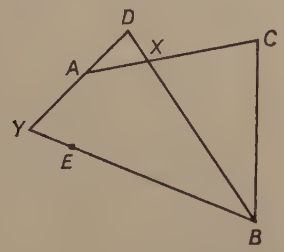


FIG. 26

Theorem. The angle in a semicircle is a right-angle.

This theorem may be applied to any diagram which contains a diameter of a circle and a point on the circle. Thus, if AB is known to be a diameter of a circle and E is any point on the circle, we apply the theorem by saying:

$A\hat{E}B = 1$ rt.-angle (angle in a semicircle).

The converse of this theorem may also be freely used in a diagram containing a known right-angle. Thus, if $A\hat{B}C$ is known to be a right-angle we can immediately say that AC is a diameter of the circle ABC (i.e. of the circle passing through A, B and C). The circle must always be named, for it means nothing to say, "$A\hat{B}C$ is a right-angle $\therefore$ AC is a diameter" (without naming the circle).

EXERCISE 8H

1. Fig. 27 shows a circle and BC, a diameter. E and F are points on the circle. BF and CE meet at A. BE and CF meet at H.

(i) Apply the theorem to BC and E.
(BC is a diameter of the circle BEC
$\therefore$ $B\hat{E}C = 90°$.)

(ii) Apply the theorem to BC and F.

(iii) Apply the converse theorem to the angle AEH.
(From (i), $B\hat{E}C$ is a right-angle
$\therefore$ $A\hat{E}H$ is a right-angle
$\therefore$ AH is a diameter of the circle AEH.)

(iv) Apply the converse theorem to angle AFC.

(v) Apply the converse theorem to angle HEC.

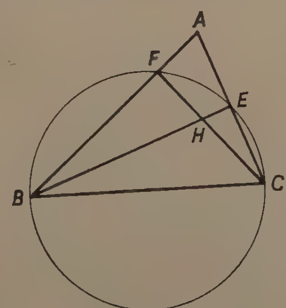


FIG. 27

2. ABC is a triangle and the circle on AB as diameter meets BC at D. Prove that AC is a diameter of the circle ADC.

3. X, Y and Z are any three points (*not* in a straight line). The circle on XY as diameter meets the circle on XZ as diameter in the points X and P. Prove that Y, P and Z are in a straight line. (Consider the angles XPY and XPZ.)

4. Apply the theorem to a diagram in which M is the mid-point of BC and A is a point such that $MA = MB$ $(= MC)$.

Proofs of the Theorems

1. The angle which an arc of a circle subtends at the centre of the circle is double the angle which the arc subtends at a point on the remaining arc of the circle.

Given: A, B and P, three points on a circle, centre C.

To prove: $A\hat{C}B = 2.A\hat{P}B$ when P is on the major arc AB; reflex $A\hat{C}B = 2.A\hat{P}B$ when P is on the minor arc AB.

Construction: Let PC meet the circle at K.

Proof: $CA = CP \quad \therefore C\hat{A}P = C\hat{P}A$

$$A\hat{C}K = C\hat{A}P + C\hat{P}A \text{ (ext. angle)}$$
$$= C\hat{P}A + C\hat{P}A \; (C\hat{A}P = C\hat{P}A)$$
$$= 2.C\hat{P}A.$$

Similarly $B\hat{C}K = 2.C\hat{P}B$.

$\therefore A\hat{C}K + B\hat{C}K = 2(C\hat{P}A + C\hat{P}B)$ and $A\hat{C}K - B\hat{C}K = 2(C\hat{P}A - C\hat{P}B)$.

Applying the first of these two statements to Fig. 28:
$A\hat{C}B = 2.A\hat{P}B$.

Applying the second of these two statements to Fig. 29:
$A\hat{C}B = 2.A\hat{P}B$.

Applying the first of these two statements to Fig. 30:
reflex $A\hat{C}B = 2.A\hat{P}B$.

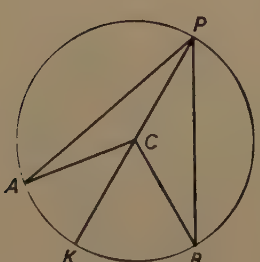


FIG. 28

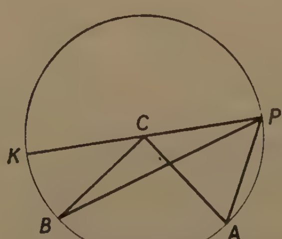


FIG. 29

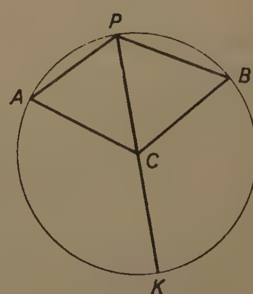


FIG. 30

2. Angles in the same segment of a circle are equal.

Given: Two angles PAQ and PBQ in the same segment.

To prove: $P\hat{A}Q = P\hat{B}Q$.

Construction: Let C be the centre of the circle (Fig. 31).

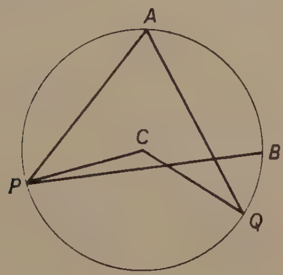


FIG. 31

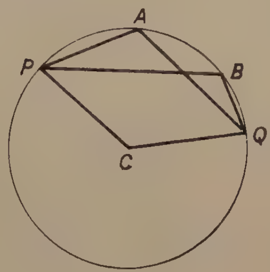


FIG. 32

Proof: If the angles PAQ and PBQ are in the major segment,

$2.P\hat{A}Q = P\hat{C}Q$ (Theorem 1)

$= 2.P\hat{B}Q$ (Theorem 1) $\therefore P\hat{A}Q = P\hat{B}Q$.

If the angles PAQ and PBQ are in the minor segment (Fig. 32),

$2.P\hat{A}Q =$ reflex $P\hat{C}Q$ (Theorem 1)

$= 2.P\hat{B}Q$ (Theorem 1) $\therefore P\hat{A}Q = P\hat{B}Q$.

3. The opposite angles of a cyclic quadrilateral are supplementary.

Given: $ABCD$ is a cyclic quadrilateral.

To prove: $B\hat{A}D + B\hat{C}D = 2$ rt.-angles.

Construction: Let E be the centre of the circle through A, B, C and D (Fig. 33).

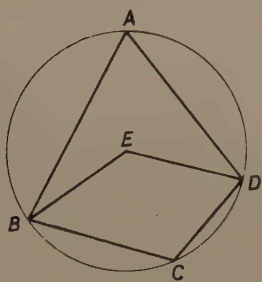


FIG. 33

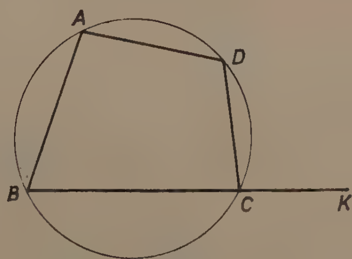


FIG. 34

Proof: (BAD is the major segment.)

$2.B\hat{A}D = B\hat{E}D$ (Theorem 1).

$2.B\hat{C}D =$ reflex $B\hat{E}D$ (Theorem 1)

$\therefore 2.B\hat{A}D + 2.B\hat{C}D = B\hat{E}D +$ reflex $B\hat{E}D = 4$ right-angles.

$\therefore B\hat{A}D + B\hat{C}D = 2$ right-angles.

4. The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

Given: A cyclic quadrilateral $ABCD$ with BC produced to K (Fig. 34).

To prove: $K\hat{C}D = B\hat{A}D$.

Proof: $K\hat{C}D = 180° - B\hat{C}D$
$= B\hat{A}D$ (Theorem 3)
$\therefore K\hat{C}D = B\hat{A}D$.

5. The angle in a semicircle is a right-angle.

Given: PQ is a diameter of a circle. A is a point on the circle.

To prove: $P\hat{A}Q = 1$ right-angle.

Construction: Let E be the centre of the circle (Fig. 35).

Proof: $2\ P\hat{A}Q = P\hat{E}Q$ (Theorem 1)
$= 2$ right-angles (since PQ passes through E)
$\therefore P\hat{A}Q = 1$ right-angle.

Note that Theorem 1 is required in the proofs of the other theorems (with the exception of 4), and if the proof of one of the Theorems 2, 3 or 5 is required, Theorem 1 should be proved, and not merely stated, as we have done here.

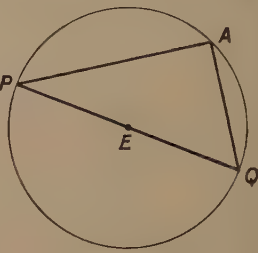


FIG. 35

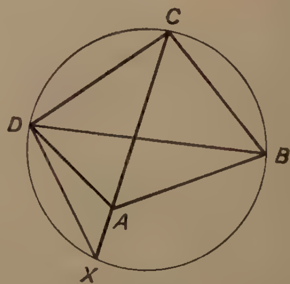


FIG. 36

6. (*Converse of* 2.) If one side of a quadrilateral subtends equal angles at the ends of the opposite side, then the quadrilateral is cyclic.

Given: A quadrilateral $ABCD$ in which $A\hat{C}B = A\hat{D}B$.

To prove: $ABCD$ is cyclic.

Construction: Let X (Fig. 36) be the point at which CA comes out of the circle BCD (we regard CA as going in at C).

Proof: $BCDX$ are concyclic (Construction).
$\therefore X\hat{D}B = X\hat{C}B$ (Theorem 2)
$= A\hat{C}B$
$= A\hat{D}B$ (Given)
$\therefore X$ lies on the line DA.
But X lies on CA (Construction)
$\therefore X$ and A have the same position.
$\therefore ABCD$ is cyclic (since $XBCD$ are concyclic by construction, and A is at X).

A word of explanation may help to understand this proof. It must be realized that X and A are described differently. X is described in the

construction and A is one of the four given points. The proof shows that the point X (described as "the point where CA comes out of the circle") has the same position as A, the given point; and since X is known to lie on the circle BCD, it follows that A lies on this circle also. This idea will be used again in proving the Converse of 3 and the Converse of 4.

7. (*Converse of* 3.) If the opposite angles of a quadrilateral are supplementary then the quadrilateral is cyclic.

Given: A quadrilateral $ABCD$ in which $B\hat{A}D + B\hat{C}D = 2$ rt.-angles.

To prove: $ABCD$ is cyclic.

Construction: Let X (Fig. 37) be the point in which CD comes out of the circle ABC (we regard CD as going in at C).

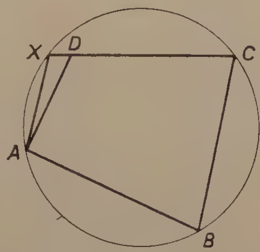


FIG. 37

Proof: $ABCX$ is cyclic (Construction)

$\therefore B\hat{A}X + B\hat{C}X = 2$ rt.-angles (Theorem 3)

$\therefore B\hat{A}X + B\hat{C}D = 2$ rt.-angles

but $B\hat{A}D + B\hat{C}D = 2$ rt.-angles (given)

$\therefore B\hat{A}D = B\hat{A}X$

$\therefore X$ lies on the line AD; but X lies on the line CD (Construction)

$\therefore X$ and D have the same position

$\therefore ABCD$ is cyclic (since $ABCX$ is cyclic).

8. (*Converse of* 4.) If an exterior angle of a quadrilateral is equal to the interior opposite angle, then the quadrilateral is cyclic.

Given: A quadrilateral $ABCD$ with BC produced to E, in which $D\hat{C}E = B\hat{A}D$.

To prove: $ABCD$ is cyclic.

Proof: $B\hat{A}D + B\hat{C}D = D\hat{C}E + B\hat{C}D$ ($B\hat{A}D = D\hat{C}E$, given)

$= 2$ rt.-angles

$\therefore ABCD$ is cyclic (Theorem 7).

The Constant Angle Locus

When a point (P) is partly located by the fact that two fixed points (A and B) subtend at P an angle of given size, the locus of P (from this fact) is two arcs of two equal circles. Fig. 38 shows the fixed points A and B and two angles, ACB and ADB of the given size. P is partly located by the fact that $APB = ACB$. We require the locus of P.

If P is on the same side of AB as C, and $A\hat{P}B = A\hat{C}B$, then A, B, C and P are concyclic. Hence the arc ACB of the circle through A, B and C is part of

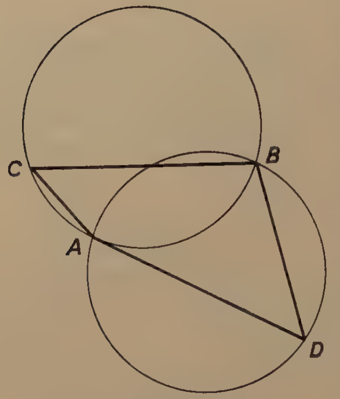


FIG. 38

the locus of P. Similarly the arc ADB of the circle through A, B and D is another part of the locus. The complete locus is these two arcs.

An angle, such as $A\hat{P}B$, which can have any number of positions, but which has a given size, is called a *constant* angle, and the result just given is referred to as the *constant angle locus.* (Strictly speaking, $A\hat{Q}B$ should be called an *angle of constant size.*)

The word "constant" should be used very carefully. It indicates that we are dealing with the members of a set of (geometrical) figures in which a certain measurement has the same value in all the figures. Thus, if we talk of a "square of constant side," we mean a member of the set of squares with sides of a given length, and when we say that the "area of a square of constant side is also constant," we mean that all members of this set have equal areas; a "triangle with base AB and constant area" means a member of the set of triangles with a given base AB and with a given area; a constant angle is thus a member of the set of angles with a given size. A circle of (or with) constant radius is a member of the set of circles which have radii of a given size. "The radius of a circle is constant" is a statement which means that a (given) circle has many radii, and these radii form a set (of lines) of which all members have the same length.

A special case of the Constant Angle Locus arises when the constant angle ($A\hat{P}B$) is a right-angle. The two arcs of the locus now belong to the same circle, namely the circle with AB as diameter. This result is worth learning on its own. It is applied by saying: $A\hat{P}B = 1$ rt. $\angle$ $\therefore$ the locus of P is the circle with diameter AB.

Exercise: ABC is an equilateral triangle of side 3 inches. Plot a point P from the facts (1) $CP = 2$ in. (2) $A\hat{P}B = 90°$.

Plot also Q from the facts (1) $\triangle\ ACQ = 3$ sq. in. (2) $B\hat{Q}C = 90°$.

Theorems concerning Chords of a Circle and their Mid-points

Theorem 1. The line joining the centre of a circle to the mid-point of a chord is perpendicular to the chord.

Theorem 2. The perpendicular from the centre of a circle to a chord bisects the chord.

Theorem 3. The perpendicular bisector of a chord of a circle passes through the centre of the circle.

Each of these three theorems is concerned with a given circle and a given chord, and with a line possessing two given properties and one deduced property, i.e. three properties in all. The three properties of the line are:

(a) It is perpendicular to the chord.

(b) It passes through the mid-point of the chord.

(c) It passes through the centre of the circle.

Each theorem states that if a line is given two of these properties, we may deduce that it has the third property. We apply the theorems by first writing the two given properties, and then the deduced property (after the sign $\therefore$).

Thus, in a diagram containing a chord, AB, of a circle centre O, and M, the mid-point of AB, we could apply Theorem 1 and write:

M is the mid-point of AB. O is the centre of the circle.

$\therefore$ OM is perpendicular to AB. (The deduced fact could be stated otherwise, e.g. $O\hat{M}A$ is a right-angle.)

In the following examples the three chord theorems (and any previous theorems) may be freely used.

EXERCISE 8J

1. M is the mid-point of a chord AB of a circle centre O. Prove that $OA^2 = OM^2 + MA^2$.

2. AB and CD are parallel chords of a circle centre O. M is the mid-point of AB. Prove that OM bisects CD.

3. Two circles with centres E and F intersect at A and B. M is the mid-point of AB. Prove that E, F and M lie in a straight line. (The proof when E and F are on the same side of AB will be different from the proof when E and F are on opposite sides of AB. Both figures and proofs should be given.)

4. A chord (AB) of a circle of radius 13 cm is 24 cm long. Calculate the (perpendicular) distance from the centre (O) to the chord.

5. A is a point in the chord QR of a circle centre O. Prove that the circle on OA as diameter passes through the mid-point of QR.

6. From Question 5 deduce the pattern formed by the mid-points of all chords, of a circle centre O, which pass through a given point A inside the circle.

7. Two parallel chords of a circle of radius 5 inches are 8 inches and 6 inches long respectively. Calculate the distance between the chords. (There are two possible answers.)

8. A circle centre O meets a line at A and B. A concentric circle (i.e. one with the same centre, O) meets the line at P and Q. Prove that $AP = BQ$. (Construction: Let OM be the perpendicular from O to the line.)

9. AB is a chord, 18 cm long, of a circle centre O and radius 12 cm. The circle centre O and radius 8 cm meets AB at P and Q. Calculate the length of PQ. (Construction: Let M be the mid-point of AB.)

Proofs of the Chord Theorems

Theorem 1.

Given: M is the mid-point of the chord AB of a circle centre O (Fig. 39).

To prove: OM is perpendicular to AB.

Proof: In the $\triangle$s OMA, OMB

$OA = OB$ (radii of a circle)

$MA = MB$ (M mid-point of AB)

$OM = OM$

$\therefore \triangle OMA \equiv \triangle OMB \therefore O\hat{M}A = O\hat{M}B$

$\therefore O\hat{M}A$ is a right-angle.

$\therefore OM$ is perpendicular to AB.

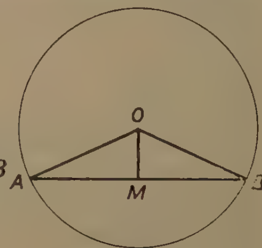


FIG. 39

Theorem 2.

Given: AB is a chord of a circle centre O and OC is the perpendicular from O to AB (Fig. 40).

To prove: C is the mid-point of AB.

Proof: In the right-angled triangles OAC, OBC

hyp. OA = hyp. OB (radii of a circle)

$OC = OC$

$\therefore \triangle OAC \equiv \triangle OBC \therefore AC = BC$

$\therefore C$ is the mid-point of AB.

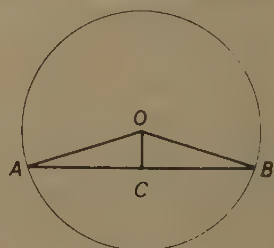


FIG. 40

Theorem 3.

Given: AB is a chord of a circle centre O.

To prove: The perpendicular bisector of AB passes through O.

Construction: Let M be the mid-point of AB.

Proof: M is the mid-point of the chord AB.

O is the centre of the circle.

$\therefore OM$ is perpendicular to AB (Theorem 1).

$\therefore OM$ is the perpendicular bisector of AB.

$\therefore$ The perpendicular bisector of AB passes through O.

(The figure for Theorem 1 can also be used for 3.)

REVISION SUMMARY

GEOMETRY

POINTS AND THEIR LOCATION

Page

To *locate* a point is to give sufficient facts to *plot* the point, i.e. to know where it is, or to mark its position.
These facts make up the *location* of the point. The locus of a (partially located) point is the pattern formed by possible positions which it (the point) can occupy.
Any fact which partially locates a point determines a locus for the point, e.g. (A, B and C are taken as completely located, i.e. fixed, points).

Fact	*Locus (from that fact)*
$AP = 5$ in	Locus of P is the circle, centre A, radius 5 in.
$A\hat{B}Q =$ rt.-angle	Locus of Q is the line through B perpendicular to AB.
$AR = CR$	Locus of R is the perp. bisector of AC.
S is 4 in from (the line) AB	Locus of S is the pair of lines parallel to AB and 4 in from AB.

Plot points from the given locations:

Point	*Location*
A	None (i.e. wherever you like)
B	$AB = 8{\cdot}4$ cm
C	$AC = 6{\cdot}8$ cm, $BC = 4{\cdot}0$ cm

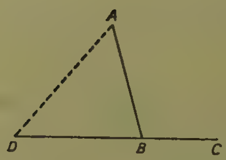
Now construct the perpendicular bisector of AC. 297
Now construct the point equidistant from A, B and C. 298
Now draw the circle passing through A, B and C. 298
(If the instructions are followed carefully, the radius of this circle should be $4{\cdot}25$ cm.)

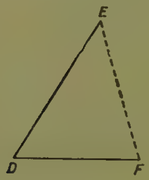
ANGLES OF A TRIANGLE

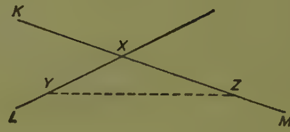
(a) An exterior angle of a triangle is equal to the sum of the interior angles at the other two vertices; and
(b) The sum of the three angles of a triangle is two right-angles.

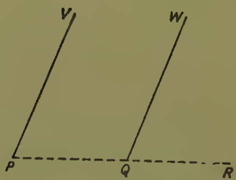
[Note: *If you are asked to prove* (b), *you must prove both* (a) *and* (b). *On no account must* (b) *be used in proving* (a).]

Page









In these figures calculate the angles ABC, D, KXY and the angle between PV and QW, given

$$A\hat{D}B = 56°,\ D\hat{A}B = 77°;$$
$$\hat{E} = k°,\ \hat{F} = m°;$$
$$L\hat{Y}Z = 147°,\ M\hat{Z}Y = 162°;$$
$$\hat{P} = 74°,\ W\hat{Q}R = 76°.$$ 305

ABC is a triangle in which $\hat{A} = 38°$ and $AB = AC$. The bisector of angle B meets AC at E. Calculate $B\hat{E}C$. (*First calculate ABC, then ABE and finally BEC.*) 308

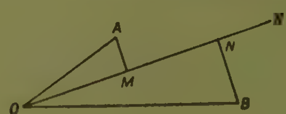
Repeat with $A = t°$ instead of 38°.

SHAPE AND SIMILARITY

Two triangles are said to be similar when each shape-item of one has the same value as the corresponding shape-item of the other. The angles of a triangle and the ratios of its sides are its shape-items.

To prove that two triangles are similar it is sufficient to show that they agree about two shape-items, but the two shape-items must be either two angles, or one angle and the ratio of the two sides which form that angle, or two ratios.

OX bisects the angle AOB. AM and BN are the perpendiculars from A and B to OX.



Prove that $\dfrac{OA}{AM} = \dfrac{OB}{BN}$. 312

ABC is a triangle. PQR is another triangle such that $\dfrac{AB}{AC} = \dfrac{PQ}{PR}$ and $\dfrac{BC}{AB} = \dfrac{QR}{PQ}$. Prove that $\hat{Q} = \hat{R}$. 313

If $\dfrac{XY}{AB} = p$ and $\dfrac{CD}{XY} = q$, what is the value of $\dfrac{CD}{AB}$? 315

Given $KL = 8$, $KM = 15$ and $L\hat{K}M$ is a right-angle, calculate LM. 319

If KL is now produced to N, making $KN = 2KL$, calculate MN, giving the answer as a surd. 320

Give the answer also as an approximation correct to three significant figures.

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AREA OF A TRIANGLE

The area of a triangle is half the area of a rectangle having the same base and height (as the triangle).

A point in a side of a rectangle forms with the ends of the opposite side (of the rectangle) a triangle whose area is half the area of the rectangle.

Two triangles with a common base are equal in area if the line joining their vertices is parallel to their (common) base.

A, B, P and Q are four points such that AB is parallel to PQ. Give the names of two pairs of equivalent triangles (i.e. triangles equal in area). 326

ABC is a triangle and a line parallel to BC cuts AB at X and AC at Y. Prove that $\triangle XYB = \triangle XYC$ and find a triangle equal in area to AXC. 326

O is the mid-point of AB. X is any other point.
Prove that $\triangle AOX = \triangle BOX$. 327

X is a point on a median, AD, of the triangle ABC. Apply the theorem "*the area of a triangle is bisected by a median*" twice to prove that $\triangle AXB = \triangle AXC$. 327

CONSTANT AREA LOCUS

If a point, P, is partially located by the fact that $\triangle BCP = \triangle BCA$, then the locus of P from this fact is a pair of lines parallel to BC, one of which passes through A and the other at an equal distance on the other side of BC.

Q and R are two given points 4 in apart. Plot P from the facts (1) $\triangle PQR = 6$ in^2; (2) $PQ = 5$ in. 328

Mark three points, A, B and C such that $AB = 3$ in and C is the mid-point of AB. Plot the point X from the location $B\hat{A}X = 37°$ and $\triangle BCX = 1{\cdot}8$ in^2. Measure AX. It should be $3{\cdot}99$ in. 328

Given a quadrilateral $ABCD$, draw a triangle ABX having the same area as the quadrilateral. 330

Reduce a pentagon to a triangle of equal area. 331

$A, B, C, \ldots I, J, K$, are eleven points on a line at distances of 1 cm apart, i.e. $AB = BC = \ldots = JK = 1$ cm. X is any other point not on the line. Give equivalent ratios for $\dfrac{\triangle XAB}{\triangle XCE}$, $\dfrac{\triangle XAD}{\triangle XBK}$, $\dfrac{\triangle XEF}{\triangle XAK}$.

Give equivalent ratios of triangles for $\dfrac{HK}{AB}$, $\dfrac{CG}{AD}$, $\dfrac{AF}{CG}$.

Give also the numerical equivalents of these ratios. 332

Page

PQR is a triangle. S is the point in PR such that $\frac{PS}{SR} = \frac{2}{1}$ and T is the point in QS such that $\frac{QT}{TS} = \frac{3}{2}$. Find numerical values for $\frac{PS}{PR}$ and $\frac{QT}{QS}$. Using these values find numerical values for $\frac{\triangle PQS}{\triangle PQR}$, $\frac{\triangle PQT}{\triangle PQS}$ and $\frac{\triangle PQT}{\triangle PQR}$ $\left(\frac{PQT}{PQR} = \frac{2}{5}\right)$. 333

A line parallel to KM meets LK at S and LM at T. Apply the theorem "*The ratio of the areas of two similar triangles is equal to the ratio of the squares of corresponding sides*" to the similar triangles LST and LKM. 334

POLYGONS

The sum of the angles of an n-sided polygon (or n-gon) is $(2n - 4)$ right-angles.

The sum of the exterior angles of any polygon is 4 right-angles.

What is the size of each angle of a regular 24-gon? 336

QUADRILATERALS

Definitions

A trapezium is a quadrilateral with a pair of parallel sides.

A parallelogram is a quadrilateral with two pairs of parallel sides.

A rhombus is a parallelogram with two adjacent sides equal.

A rectangle is a parallelogram with one angle a right-angle.

A square is a rhombus with one angle a right-angle—or a rectangle with two adjacent sides equal.

Theorems about Parallelograms.

The opposite sides of a parallelogram are equal.

What are the given properties and what are the deduced properties in this theorem? 343

A quadrilateral with both pairs of opposite sides equal is a parallelogram.

What are the given properties and what are the deduced properties in this theorem? 344

A quadrilateral with one pair of sides equal and parallel is a parallelogram.

What are the given, and what the deduced properties in this theorem? 344

The diagonals of a parallelogram bisect one another.

Rectangles, rhombi and squares have all the properties of parallelograms, and they have additional properties of their own.

The diagonals of a rectangle are equal.

Each angle of a rectangle is a right-angle.

Page

All four sides of a rhombus are equal in length.
The diagonals of a rhombus meet at right-angles.
The diagonals of a rhombus bisect the angles of the rhombus.

Note that all these theorems are also true about squares.

Given that $ABCD$ is a parallelogram, that G is the mid-point of AB and that H is the mid-point of CD, prove that AH is parallel to GC. 366

Given that M is the mid-point of AB and that $AMXY$ is a parallelogram, prove that MY is parallel to BX. 367

$ABCD$ is a quadrilateral in which $A\hat{C}B = A\hat{D}B$. What kind of a quadrilateral is $ABCD$? (*Give your answer in the form*: $A\hat{C}B = A\hat{D}B$. $\therefore$ *ABCD is a . . . quadrilateral.* $\therefore \ldots \hat{.} \ldots = \ldots \hat{.} \ldots$ *and* $\ldots \hat{.} \ldots = \ldots \hat{.} \ldots$ *There are three pairs of equal angles in the figure.*)

If, in this same quadrilateral $ABCD$, AC and BD meet at O, what can be deduced about OA, OB, OC and OD? 350

$ABCD$ is a quadrilateral in which $A\hat{D}B = A\hat{C}B$ and AC bisects $B\hat{A}D$. Prove that $B\hat{D}C = D\hat{B}C$. 351

The opposite angles, $\hat{P}$ and $\hat{R}$ of the quadrilateral $PQRS$ are right-angles. What kind of a quadrilateral is $PQRS$? Deduce that $Q\hat{P}R = Q\hat{S}R$. (*Give your answer in the form*: *PQRS is a . . . quad.* $\therefore$ *etc.*)

CONGRUENT TRIANGLES

Two triangles are said to be congruent when each is an exact copy of the other.
Each side and each angle of either triangle has an equal counterpart in the other.
If two triangles agree about

1. **three sides**
2. **two sides and the included angle**
3. **one side and the angles at its ends,**

then the triangles are congruent.

Note: (*i*) *Agreement about the three angles is not sufficient to deduce that two triangles are congruent.*
(*ii*) *Agreement about two sides and a not-included angle is not sufficient to deduce that two triangles are congruent.*

If two right-angled triangles agree about the hypotenuse and about one other side, then the triangles are congruent.

AB and CD bisect each other at O. Prove that AC is parallel to BD. 359

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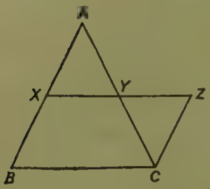
XO and OY are two perpendicular and equal lines. P is a point in OX. The circle centre X with radius YP meets OY at Q. Prove that $OP = OQ$. 358



In this figure Y is the mid-point of AC, XYZ is parallel to BC and CZ is parallel to BXA. Prove that $AX = XB$ 361

STATEMENT, APPLICATION AND PROOF OF A THEOREM

The following are statements of well-known and often-used theorems. Every theorem has "given" properties and "deduced" properties. These statements are arranged so that the given properties follow the word "if" and the deduced properties follow the word "then".

If two sides of a triangle are equal (to one another) then the angles opposite those sides are equal (to one another). If two angles of a triangle are equal, then the sides opposite those angles are equal.

Note that in these two theorems, the given and deduced facts are interchanged. Such pairs of theorems are called "converse".

If a line bisects one side of a triangle and it is parallel to a second side, then it bisects the third side.

If a line bisects two sides of a triangle, then it is parallel to the third side.

If a line joins the mid-points of two sides of a triangle, then it is half the length of the third side.

If a quadrilateral is a parallelogram, then
its opposite sides are equal,
its opposite angles are equal,
its diagonals bisect one another.

(*Three theorems with the same given property.*)

State the converses of these three theorems. 366

M is the mid-point of the hypotenuse, AB, of the right-angled triangle ABC. N is the mid-point of AC.

Apply the first Mid-point Theorem to this diagram. 369

Deduce that MN is perpendicular to AC (i.e. M is on the perpendicular bisector of AC).

Deduce that $MA = MB = MC$. 370

What can be said about the point M and the circle passing through A, B and C? 404

If three parallel lines make equal intercepts on a straight line, then they make equal intercepts on any other line.

CIRCLES

	Page
What is the difference in meaning between the phrases "the radius of a circle" and "a radius of a circle"?	390
Of all the secants which cut a given circle at a point A, what name is given to the one with the longest chord? What name is given to that with the shortest chord?	390
What name is given to a figure bounded by an arc of a circle and the chord joining the ends of the arc?	391
Given a segment of a circle, how would you draw an "angle in that segment"? (*Answer this question by stating how the vertex and arms of the angle would be chosen.*)	391
What is the size of an angle in a semicircle?	391
Apply this result to a diagram in which AB is a diameter of a circle and P is a point on the circle. Give the name of the angle "subtended by" AB at P.	
What is the size of the angle subtended by a side of a square at either of the other vertices (of the square)?	
An arc of a circle subtends an angle of 108° at the centre of the circle. What angles does it subtend at points on the circle?	393
$ABXD$ is a quadrilateral in which $A\hat{B}X = A\hat{D}X$. AB and DX, produced, meet at E. AD and BX, produced, meet at C. Prove that B, C, D and E are concyclic.	398
State theorems which mention:	
The angle at the centre of a circle.	402
Angles in the same segment of a circle.	403
The exterior angle of a cyclic quadrilateral.	404
The opposite angles of a cyclic quadrilateral.	403
O is a point inside the triangle ABC. X, Y and Z are the feet of the perpendiculars from O to BC, CA and AB respectively. Name three sets of concyclic points (four points in each set). Name also a diameter of each of the three circles (on which the concyclic sets lie). Give a reason for each statement you make.	404
Given a chord (AB) of a circle (centre O), what other property may be deduced of a line which:	
(*a*) Passes through the centre and bisects the chord.	406
(*b*) Passes through the centre and is perpendicular to the chord.	406
(*c*) Bisects the chord and is perpendicular to the chord.	407

ANSWERS TO EXERCISES

ARITHMETIC

Exercise 1A

1. $\frac{2}{3}$
2. $\frac{4}{5}$
3. $\frac{1}{2}$
4. $\frac{1}{3}$
5. $\frac{4}{5}$
6. $\frac{3}{5}$
7. $\frac{4}{5}$
8. $\frac{7}{11}$
9. $\frac{2}{3}$
10. $\frac{2}{9}$

Exercise 1B

1. Yes
2. No
3. Yes
4. 2, 3, 4, 6
5. No
6. Yes
7. Yes

Exercise 1C

1. (a) 8 (b) 9 (c) 25 (d) 16 (e) 32 (f) 24 (g) 108 (h) 600
2. 2×9; 3×6; $\underline{2 \times 3 \times 3}$ $(= 2 \times 3^2)$
3. $2^3 \times 3$
4. 4

Exercise 1D

1. $2^3 \times 3 \times 5 \times 7$
2. $2^2 \times 7 \times 13$
3. $3^2 \times 17$
4. $2 \times 3 \times 13 \times 17$
5. $2^5 \times 3^3$

Exercise 1E

1. $\frac{13}{30}$
2. $\frac{17}{96}$
3. $\frac{144}{221}$
4. $\frac{35}{36}$

Exercise 1F

1. 6
2. 34
3. 15
4. 44
5. 25
6. 27 in.

Exercise 1G

1. $\frac{5}{12}$
2. $\frac{3}{20}$
3. $\frac{2}{5}$
4. $\frac{5}{6}$
5. $1\frac{1}{10}$
6. $\frac{10}{23}$
7. $1\frac{1}{2}$
8. 6
9. $1\frac{1}{2}$
10. 1

Exercise 2A

1. 26 ft
2. 460p
3. 76 cwt
4. 4 ft
5. 3 gal 2 pt
6. £1·28
7. $2\frac{1}{2}$ lb
8. 4 min 10 s

Exercise 2B

1. $\frac{1}{3}$ **2.** $\frac{1}{4}$ **3.** $\frac{2}{5}$ **4.** $\frac{1}{8}$ **5.** $\frac{2}{7}$ **6.** $\frac{1}{16}$

Exercise 2C

1. 18 in² **2.** 40 ft² **3.** 6 yd² **4.** 120 mile² **5.** 168 in² **6.** 12 ft² **7.** 540 in²

Exercise 2D

1. 2 in **2.** 6 ft **3.** 27 yd **4.** 14 ft **5.** 15 in **6.** 30 in

Exercise 2E

1. 7 in² **2.** 18 ft² **3.** 12 ft²

Exercise 2F

1. 87 ft² **2.** 468 ft² **3.** 18 ft²

Exercise 2G

1. 16 in **2.** 38 ft **3.** 26 ft

Exercise 2H

1. 576 ft² **2.** 105 ft² **3.** 164 in²

Exercise 2J

1. 72 ft³ **2.** 60 yd³ **3.** 144 ft³ **4.** 320 in³; 18 **5.** 2 in

Exercise 2K

1. 40 in³ **2.** 1200 in³ **3.** 288 in³

Exercise 3A

1. 64p **2.** 750 miles **3.** £1·35 **4.** 3 lb. **5.** 25p **6.** 48 sec.

Exercise 3B

1. £34·20 **2.** 3 miles; 12 miles **3.** £152; £456 **4.** 48

5. 5 s: 450 s
6. 32 min
7. 75 min
8. £4·95
9. £7·80
10. 36 lb
11. 500
12. £2·50

Exercise 3C

1. $2\frac{1}{2}$ hours
2. 64
3. 9 hours
4. 4
5. 10 in
6. 42
7. 5 days
8. 100 s
9. $1\frac{2}{3}$ hours; $3\frac{1}{3}$ hours
10. 15 days

Exercise 3D

1. 28 weeks
2. 25 days
3. No answer possible
4. No answer
5. £33
6. No answer
7. 66 ft. per sec.; $66n$ ft. per sec.
8. No answer
9. 8.20 p.m.
10. 350
11. 270
12. 5 minutes (at the same rate!)
13. Yes; $7\frac{1}{5}$ hours
14. $19\frac{1}{5}$ hours
15. 32 hours; $\dfrac{96}{x}$ hours

Exercise 3E

1. 9 mile/h
2. 7p per lb
3. 7p per ft
4. 60 ft/s
5. 12 000 mile/h, or 200 mile/min
6. 32 pupils per teacher
7. 13 runs per wicket
8. $9\frac{1}{2}$p per lb
9. 35 mile/h
10. $35\frac{1}{3}$ mile/h
11. 88 ft/s; 44 ft/s; 66 ft/s
12. 18; 16; Taylor
13. 61p an hour; £28·06
14. $1\frac{1}{2}$ in per yd
15. 10 mile/h
16. 9 a.m.

Exercise 3F

1. 4 days 11 hours
2. 3.30 p.m.
3. 165 miles
4. 18 480 yd
5. 10 min; 30 min; 40 min; 3 mile/h
6. 7 hours
7. 30 mile/h
8. 11p
9. $12\frac{2}{3}$ mile/h; 384 ft; $7\frac{2}{5}$ hours; 149 knots; $4\frac{8}{9}$ hours; 2 miles

Exercise 4A

1.

	Hundreds 100	*Tens* 10	*Units* 1	*Tenths* $\frac{1}{10}$	*Hundredths* $\frac{1}{100}$	*Thousandths* $\frac{1}{1000}$
$\frac{3}{10}$	—	—	—	3	—	—
$\frac{7}{100}$	—	—	—	—	7	—
$\frac{86}{100}$	—	—	—	8	6	—
$2\frac{1}{10}$	—	—	2	1	—	—
5·94	—	—	5	9	4	—
78·12	—	7	8	1	2	—
935·307	9	3	5	3	0	7
nine thousandths	—	—	—	—	—	9
twelve hundredths	—	—	—	1	2	—

2. 1·3; 5·7; 134·1; 0·6; 3·2; 12·23; 9·002; 2·07; 67·034; 1·203
3. $5\frac{1}{10}$; $3\frac{3}{10}$; $7\frac{9}{10}$; $\frac{1}{2}$; $4\frac{4}{5}$; $4\frac{9}{50}$; $18\frac{33}{100}$; $1\frac{5}{8}$
5. 0·027; 0·424; 3·2; 6·21; 1·007; 0·36; 14·278
7. $\frac{1}{50}$; $\frac{2}{5}$; $\frac{1}{4}$; $\frac{3}{4}$; $\frac{1}{8}$; $\frac{18}{25}$; $\frac{7}{8}$
8. 5·1 in.
9. $\frac{5}{10}$; 0·5; $\frac{6}{10}$; 0·6; $\frac{8}{10}$; 0·8; $\frac{2}{10}$; 0·2
10. $\frac{4}{100}$; 0·04; $\frac{12}{100}$; 0·12; $\frac{68}{100}$; 0·68; $\frac{5}{100}$; 0·05; $\frac{65}{100}$; 0·65
12. fr. 1, 72c; fr 34, 7c; fr. 12, 96c
13. 1·7 miles West
14. 36·34
15. 28·82
16. 149·728
17. 84·06
18. 5·407
20. 9, 10, 27

Exercise 4B

1. 40·68
2. 28·96
3. 22·95
4. 26·97
5. 51·85
6. 145·84
7. 99·442
8. 56·799
9. 150·870
10. 21·200
11. 153·16
12. 154·822
13. 12·4
14. 58·5
15. 3·5
16. 6·77
17. 41·52
18. 45·87
19. 4·38
20. 191·351

Exercise 4C

1. 0·2	**4.** 12·835	**7.** 3·05 in.	**10.** 5·84 in.
2. 4·52	**5.** 19·363	**8.** 2·6 in.	
3. 13·297	**6.** 20·1° C.	**9.** 14·12 in.	

Exercise 4D

1.	(a) 3·2	(b) 3·17	(c) 3	
2.	(a) 5·0	(b) 5·04	(c) 5	
3.	(a) 8·9	(b) 8·89	(c) 9	
4.	(a) 1·0	(b) 1·05	(c) 1	
5.	(a) 0·6(5)	(b) 0·65	(c) 1	
6.	(a) 38·5	(b) 38·53	(c) 39	
7.	(a) 15·2(5)	(b) 15·25	(c) 15	
8.	(a) 10·0	(b) 10·00	(c) 10	
9.	(a) 3·4318	(b) 3·432	(c) 3·43	
10.	(a) 2·0832	(b) 2·083	(c) 2·08	(d) 3·4
11.	(a) 1·0074	(b) 1·007	(c) 1·01	(d) 2·1
12.	(a) 2·9739	(b) 2·974	(c) 2·97	(d) 1·0
13.	(a) 23·1309	(b) 23·131	(c) 23·13	(d) 3·0
14.	5·423			

Exercise 4E

1.	(a) 32	(b) 320	(c) 3200
2.	(a) 43·8	(b) 438	(c) 4380
3.	(a) 7·6	(b) 76	(c) 760
4.	(a) 0·25	(b) 2·5	(c) 250
5.	(a) 150·03	(b) 1500·3	(c) 15 003
6.	(a) 7	(b) 0·7	(c) 0·07
7.	(a) 42·56	(b) 4·256	(c) 0·4256
8.	(a) 0·1234	(b) 0·012 34	(c) 0·001 234
9.	(a) 0·005	(b) 0·0005	(c) 0·000 05
10.	(a) 5·837	(b) 0·5837	(c) 0·058 37

11. 10	**15.** 1000	**19.** 0·0082 in
12. 100	**16.** 10	**20.** (a) 6
13. 10	**17.** 1 in	(b) 1·2
14. 1000	**18.** $2\frac{1}{2}$	(c) 30

Exercise 4F

1. 12·8	**6.** 6·328	**11.** 0·024	**16.** 300
2. 34·2	**7.** 162·12	**12.** 1·08	**17.** 34
3. 75·4	**8.** 44·088	**13.** 14·4	**18.** 77
4. 111·06	**9.** 62·72	**14.** 12	**19.** 763
5. 0·057	**10.** 24·6036	**15.** 30	**20.** 295·2

Exercise 4G

1. 2·7	**4.** 3·43	**7.** 3·5	**10.** 0·0793	**13.** 0·5
2. 2·9	**5.** 0·084	**8.** 0·013	**11.** 13·334	**14.** 3·75
3. 0·09	**6.** 0·35	**9.** 0·85	**12.** 0·05	**15.** 0·0625

Exercise 4H

1. 0·08	**6.** 0·020	**11.** 109·8632	**16.** 1965·4
2. 0·24	**7.** 3·35	**12.** 16·6448	**17.** 27·726 24
3. 3·2	**8.** 8·213	**13.** 14·8	**18.** 19·6726
4. 2·24	**9.** 7·632	**14.** 468	**19.** 0·001 287 5
5. 1·62	**10.** 447·5947	**15.** 7·4572	**20.** 75·210 588

Exercise 4J

1. 40	**5.** 31·4	**9.** 4·5	**13.** 8·53	**17.** 53
2. 900	**6.** 5780	**10.** 7·8	**14.** 0·001	**18.** 107
3. 900	**7.** 543	**11.** 54 000	**15.** 0·32	**19.** 3·04
4. 2·4	**8.** 0·0814	**12.** 26	**16.** 0·47	**20.** 30

Exercise 4K

1. 0·333	**3.** 0·111	**5.** 0·555	**7.** 0·428(5)	**9.** 0·454(5)
2. 0·833	**4.** 0·222	**6.** 0·143	**8.** 0·571	**10.** 0·538

Exercise 5A

1. 6	**3.** 18	**5.** 60	**7.** 24	**9.** 60
2. 12	**4.** 12	**6.** 60	**8.** 20	**10.** 120

Exercise 5B

1. $\frac{3}{4}$	**3.** $\frac{5}{12}$	**5.** $\frac{1}{5}$	**7.** $\frac{5}{24}$	**9.** $\frac{37}{50}$
2. $\frac{11}{12}$	**4.** $1\frac{7}{18}$	**6.** $\frac{6}{13}$	**8.** $\frac{109}{120}$	**10.** $\frac{43}{72}$

Exercise 5C

1. $5\frac{2}{3}$
2. $13\frac{5}{8}$
3. $12\frac{7}{12}$
4. $6\frac{5}{24}$
5. $2\frac{1}{8}$
6. $3\frac{3}{10}$
7. $10\frac{2}{3}$
8. $1\frac{7}{18}$
9. $14\frac{23}{24}$
10. $6\frac{97}{120}$

Exercise 5D

1. $2^3 \times 3^2 \times 5 \times 7$ (=2520): 14, 15: $\frac{1129}{2520}$
2. $3^2 \times 7 \times 13$ (= 819): 13, 3: $\frac{256}{819}$
3. $3^3 \times 5^3 \times 11 \times 7$ (= 259 875)
4. $\frac{1463}{1800}$

Exercise 5E

1. 9
2. 11
3. 2
4. 1
5. 28
6. 54
7. 81
8. 55
9. 49
10. 42

Exercise 5F

1. $1\frac{1}{40}$
2. $\frac{23}{24}$
3. $\frac{1}{10}$
4. $\frac{7}{30}$
5. 33
6. $1\frac{11}{30}$
7. $1\frac{37}{210}$
8. $\frac{3}{4}$
9. $1\frac{20}{33}$
10. $\frac{147}{548}$

Exercise 5G

1. £1·65
2. 15p
3. $2\frac{1}{4}$
4. $5\frac{1}{4}$ ft^2
5. $7\frac{1}{2}$ in

Exercise 6A

1. 100
2. 1000
3. 100 000
4. 300
5. 2000
6. 4000
7. (i) 2·04 m; (ii) 204 cm
8. (i) 5·1 m; (ii) 510 cm
9. (i) 3040 m; (ii) 3·04 km
10. (i) 6700 m; (ii) 670 000 cm
11. 0·035 km
12. 8000 mg
13. 0·762 kg
14. $\frac{1}{2}$ kg
15. About 1 mile (0·93 miles)
16. 295 miles; 520 km

Exercise 6B

1. 100; 1 000 000
2. 1 000 000
3. 0·000 001
4. 0·48
5. 0·34
6. 56
7. 5000; 700; 73
8. $5\frac{1}{4}$; $\frac{7}{16}$
9. fr.8
10. \$6·45
11. 300 cm^3; $1000x$ cm^3
12. 3·6 kg

13. 108·24 m
14. \$33·6 or \$33, 60 cents
15. 275 ml
16. 100 kg
17. 800 g or 0·8 kg
18. 15 m^3; 15 000 litres; 15 000 kg
19. 800 g or 0·8 kg
20. 1·486 km; 13·5 ha

Exercise 6C

1. 1·8 m
2. 510 km
3. 0·05 mm
4. 0·714 kg
5. 0·5 m
6. 90 000 miles
7. 64 cm
8. 0·0049 m
9. 3·01 litres
10. 5·0 g
11. 200 in
12. 3500 yd
13. 73·35—73·45 miles; 220·05—220·35 miles
14. Greatest 8·675 in; Least 8·645 in
15. (i) 4; (ii) 2 (perhaps 3)
16. 1·55 to 1·65 m; 294·5 m to 313·5 m
17. $61\frac{1}{2}$p—$64\frac{1}{2}$p
18. 2 m^2, correct to 1 figure
19. 10 cm^2, correct to 1 figure
20. 380 000, correct to 2 figures.

Exercise 6D

1. 0·190
2. 0·292
3. 0·187(5)
4. 0·389
5. 0·312(5)
6. 0·444
7. 0·150
8. 0·625
9. 0·038
10. 0·167
11. 1·575 tons
12. 4·228 hours
13. 3·375 gal
14. 2·651 miles
15. 3·944 yd
16. 1·540 quarters
17. 4·320 miles
18. 2·137(5) tons

Exercise 7A

1. $\frac{1}{5}$
2. $\frac{1}{3}$
3. $\frac{1}{8}$
4. $\frac{20}{1}$
5. $\frac{3}{5}$
6. $\frac{5}{16}$
7. $\frac{20}{3}$
8. $\frac{3}{10}$
9. $\frac{4}{1}$
10. $\frac{1}{5}$

Exercise 7B

1. 25 : 3
2. 4 : 1
3. 65 : 44
4. 1 : 19
5. 11 : 36
6. 90 : 1
7. 7 : 5
8. 1 : 500
9. 3 : 20
10. 1 : 32
11. 3 : 4; 9 : 16
12. 20 : 3; 3 : 20; 3 : 23
13. 5 : 8
14. 1 : 5280
15. 33 : 38

Exercise 7C

1. 20 ft
2. 26
3. 20 ft
4. £180
5. 91p
6. $2\frac{2}{5}$ hours; $11\frac{2}{3}$ hours
7. $4\frac{4}{5}$ km; $6\frac{1}{4}$ miles
8. 10 cm

Exercise 7D

1. £20, £12
2. 1·25 kg; 1·75 kg
3. 50 min; 1 h 40 min; 2 h 30 min
4. $26\frac{1}{4}$ yd; $43\frac{3}{4}$ yd
5. 1 : 2 : 3
6. 1 in; 3·5 in
7. 2·4 cm
8. 90, 45, 15p
9. £1·50; £1; £2
10. 400 : 210 : 1
11. 36 min
12. £2·25; £1·80; £1·50
13. 18
14. 5 : 3, 3 : 5

Exercise 7E

1. 9 : 6 : 10
2. 25 : 12
3. 64 : 99
4. 3·6 in
5. Increase in ratio 35 : 27

Exercise 7F

1. 25
2. 3 kg
3. 4 ft 6 in
4. $3\frac{1}{2}$ hours
5. 1 metre
6. 20 mile/h
7. 10 lb
8. 10 yd
9. 20 ft/s
10. 100 g
11. 48p
12. 20p
13. 30 lb
14. $52\frac{1}{2}$p
15. 300 ml
16. $1\frac{1}{2}$ gal
17. 30 minutes
18. 44·2 m
19. £680 per annum
20. 1 lb 8 oz
21. $31\frac{1}{2}$p
22. 5 : 6
23. 1 : 96
24. 4 : 5
25. 30 cm by 24 cm; 4 : 9

Exercise 7G

1. £8·80
2. 8 : 9
3. 13 : 10
4. Increased in ratio 9 : 8
5. 19 : 22; increased in ratio 42 : 41

Exercise 7H

1. 1 : 1·5
2. 1 : 1·33
3. 1 : 1·67
4. 1 : 0·375
5. 1 : 0·583
6. 1 : 4·75
7. 0·667 : 1
8. 0·75 : 1
9. 2·67 : 1
10. 12·2 : 1
11. 0·0818 : 1
12. 1·30 : 1
13. 4 : 5
14. 19 : 12
15. Same
16. 29 : 7

Exercise 7J

1. $\frac{1}{1760}$
2. $\frac{1}{31680}$
3. $\frac{1}{6336}$
4. $\frac{1}{126720}$
5. $\frac{1}{100000}$
6. $\frac{1}{200}$
7. 21·12 in
8. 12 miles
9. 72·6 km
10. 64 000 acres

Exercise 7K

1. $10\frac{2}{3}$ hours
2. $33\frac{3}{4}$ in
3. 84 minutes
4. $6\frac{1}{4}$ days
5. £100
6. $7\frac{1}{2}$ minutes
7. £450
8. $7\frac{1}{2}$ lb
9. $\dfrac{xy}{z}$
10. 6·6 kg
11. No proportional connexion
12. 2736 r.p.m.
13. 1 in
14. 136·5 miles
15. £25·76

Exercise 7L

1. £36
2. 6 days
3. 9 men
4. 500 kg
5. £157·50
6. $6\frac{2}{9}$ hours
7. £13·50
8. 28 weeks
9. 100 kg
10. $\dfrac{BXZ}{AY}$ hours
11. $5\frac{1}{3}$ days

Exercise 8A

1. $\dfrac{4}{100}$; 4%
2. $\dfrac{15}{100}$; 15%
3. $\dfrac{40}{100}$; 40%
4. $\dfrac{37\frac{1}{2}}{100}$; $37\frac{1}{2}$%
5. $\dfrac{62\frac{1}{2}}{100}$; $62\frac{1}{2}$%
6. $\dfrac{87\frac{1}{2}}{100}$; $87\frac{1}{2}$%
7. $\dfrac{35}{100}$; 35%
8. $\dfrac{11\frac{1}{9}}{100}$; $11\frac{1}{9}$%
9. $\dfrac{16\frac{2}{3}}{100}$; $16\frac{2}{3}$%
10. $\dfrac{166\frac{2}{3}}{100}$; $166\frac{2}{3}$%
11. 8%; 24
12. (a) 40%
 (b) $\frac{1}{2}$%
 (c) 16%
 (d) $87\frac{1}{2}$%
13. 5%
14. 92%
15. 72%
16. (a) $\frac{3}{10}$
 (b) $\frac{7}{10}$
 (c) $\frac{9}{10}$
 (d) $\frac{17}{20}$
 (e) $\frac{3}{20}$
 (f) $\frac{5}{8}$
17. (a) 2
 (b) $3\frac{1}{5}$
 (c) $1\frac{1}{4}$
 (d) $2\frac{9}{20}$
18. (a) 15%
 (b) 37%
 (c) 4%
 (d) 247%
 (e) 59·1%
 (f) 70·3%

Exercise 8B

1. 25%
2. $33\frac{1}{3}$%
3. $33\frac{1}{3}$%
4. $166\frac{2}{3}$%
5. 35%
6. $8\frac{1}{3}$%
7. $6\frac{1}{4}$%
8. $28\frac{3}{4}$%
9. 84%
10. 16%
11. $61\frac{1}{2}$p
12. £2·64
13. £1·$30\frac{1}{2}$
14. 14p
15. 6p
16. £17·28
17. 80
18. 36
19. £18
20. 6912
21. $11\frac{1}{9}$%
22. 51p (15% = 10% + 5%; $\frac{1}{10}$ of £3·38 = £0·338; $\frac{1}{2}$ of that (5%) = £0·169; £0·338 + £0·169 = £0·507)
23. 34 girls
24. $\dfrac{100x}{y}$%

Exercise 8C

1. 31
2. $213\frac{3}{4}$
3. 286
4. 45
5. 40
6. 300
7. 80
8. £99
9. £68
10. £59·50
11. 1 ton 13 cwt
12. 1200
13. £2·14
14. 1560
15. £900
16. £48·30
17. 195
18. 33 000
19. 5%
20. 2%

Exercise 8D

1. £10·50
2. £84
3. £100
4. $2\frac{1}{2}$p
5. 14p
6. £282·80
7. £10
8. £2·50
9. £80
10. £2·40
11. £37·50
12. 42p
13. 25%
14. 2%
15. 6%
16. $12\frac{1}{2}$%
17. 25%
18. 16%
19. $12\frac{1}{2}$%
20. $9\frac{3}{8}$%
21. $6\frac{1}{4}$%
22. $16\frac{2}{3}$%
23. £96
24. $27\frac{3}{11}$%
25. 60%
26. £800
27. £1680
28. $52\frac{1}{2}$p
29. £75
30. 21p
31. 50%
32. £28 : £33·60
33. £337·50
34. $16\frac{2}{3}$%
35. $90n$ p

Exercise 9A

1. £7 3. £16 5. £90 7. £12·50 9. £60
2. £2·50 4. £6 6. £17·50 8. £27 10. £21

Exercise 9B

1. £136 3. £427 5. £266·87$\frac{1}{2}$
2. £696 4. £1012·50 6. £1378

Exercise 9C

1. £150 2. £300 3. £280 4. £25 5. £27

Exercise 9D

1. £15
2. £32·20
3. £97·12$\frac{1}{2}$
4. £2·25
5. £5·25
6. £15·10
7. £5·46
8. £6·57
9. £1·24
10. £46·37
11. £2·44
12. 18p
13. £8·33
14. £3·38
15. £1·30
16. 50p
17. £540·50
18. £237·50
19. £4·57
20. $\dfrac{abc}{£1200}$

Exercise 9E

1. 3$\frac{1}{3}$ years
2. £340
3. 4%
4. 1 year
5. 32%
6. £80
7. 5$\frac{1}{2}$%
8. £5600
9. £50, 3$\frac{11}{13}$%
10. £38·40

Exercise 10A

1. 1 000 000 2. 1 000 000 3. 1 000 000 4. 46 656

Exercise 10B

1. 2700 in^3; 1$\frac{9}{16}$ ft^3
2. 44·2 m^2
3. 1$\frac{3}{4}$ acres
4. 1$\frac{2}{3}$ ft^3
5. 2304 acres
6. 36 ft
7. 6·5 m^2

Exercise 10C

1. £1020
2. 86·4 kg
3. 2064 in^3
4. 40
5. £4·95
6. £6·40
7. 2 m
8. 135 ft^3

Exercise 10D

1. 54 in^2
2. 30 in^2
3. 99 cm^2
4. 31·5 cm^3
5. $1026\frac{2}{3}$ yd^3
6. 20 250 ft^3

Exercise 10E

1. 250 gal; 2500 lb
2. 42 litres; 42 kg
3. 126 600 gal approx.; 565·1 tons approx.
4. 2·27 kg approx.

Exercise 10F

1. $1\frac{1}{2}$ in
2. 2·4 ft
3. 80 cm
4. 27 in
5. 6 ft
6. 14 tons; 63 min

ALGEBRA

Exercise 1A

1. 29 years, $35 - L$ years
2. $28 + x$ children, 33 children
3. $12 - b$ m; 10 m
4. $50 - C$ children
5. $6 + g$ oz
6. $6 - y$ lb, $x - y$ lb
7. $N + 2$. $N + 4$
8. $1050 + m$ miles
9. $p - 4$ pence
10. $3 + x + y$ players

Exercise 1B

1. $40t$ miles
2. $14x$ loaves
3. $15n$ stamps
4. $34x$ in
5. $15b$ m^2
6. $7x$ s; $31x$ s
7. rs pence
8. $112d$ pence
9. $16aw$ pence
10. 100 pence, 300 pence, $100b$ pence

Exercise 1C

1. $2a$
2. $5e$
3. b
4. $21x$
5. $54b$
6. $32y$
7. $3ab$
8. $16cd$
9. lmn
10. 0

Exercise 1D

1. $\frac{15}{b}$ in
2. £2; £6; £$\frac{x}{100}$
3. $\frac{30}{n}$ p
4. 2 cm; $\frac{8}{x}$ ft; $\frac{A}{4}$ m; $\frac{A}{L}$ in
5. £$\frac{A}{56}$
6. $\frac{A}{2}$ apples
7. £$\frac{r}{30}$
8. $\frac{p}{r}$ classes
9. £$\frac{V}{2w}$
10. $\frac{120}{S}$ h

Exercise 1E

1. $x + 4 = 8$
2. $2a = 3$

3. $b+4$
4. $v-4=12$
5. $y+y+5$ (or $2y+5$)
6. $\frac{m}{3}=4$
7. $3y-8x$
8. $4B=15-B$
9. $\frac{R}{2}=6 \therefore R=12$
10. $\frac{3d+7}{11}=2$
11. $3ab$
12. $a=b$
13. $\frac{W}{2}=1$
14. $\frac{1}{3}-\frac{x}{6}$
15. $\frac{m+n}{s}$

Exercise 1F

1. $3m$
2. $5v$
3. $3a$
4. b
5. 0
6. $4a$
7. $6b$
8. $3m$
9. $9d$
10. $7x$
11. $6d$
12. $9z$
13. x
14. $10V$
15. $8M$
16. 0
17. $6x$
18. No simpler form
19. No simpler form
20. $2a+b$
21. $2x+y$
22. $2d+2e$
23. $4d-2c$
24. $2c$
25. $10h+5g$
26. $5x+y$
27. $6m+6$
28. $c+11$
29. $5x-2a$
30. $3q-7r$

Exercise 1G

1. 20
2. 9
3. 10
4. 7
5. 1
6. 24
7. 0
8. 12
9. 33
10. 1
11. 2
12. 2
13. 4
14. $\frac{3}{8}$
15. 1
16. 1
17. $\frac{1}{4}$
18. $\frac{1}{12}$
19. $\frac{1}{12}$
20. $4\frac{2}{3}$

Exercise 1H

1. $\frac{a}{20}$ tons $\frac{s}{2240}$ tons
2. 50 miles, 125 miles, $25z$ miles
3. 100p, 400p, $100x$ p
4. 100 cm, $100b$ cm, $100\,000x$ cm
5. 96 oz; $16z$ oz; $8z$ oz
6. $1000m$ g
7. $3xy$ ft²
8. $n-2$
9. $\frac{2a}{7}$ weeks; $52y$ weeks
10. $2l+2m$ ft

Exercise 1J

1. 49
2. 81
3. 32
4. 3
5. 4
6. 7
7. 98
8. 1323
9. 256
10. 882
11. 2268
12. 2016

Exercise 1K

1. s^3
2. $2t^3$
3. $6ab$
4. cd^2
5. c^2d^2
6. $4a^3$
7. a^2b^2
8. xy^2z
9. $8uv^2w$
10. $6x^3y^2$

Exercise 1L

1. $4x$
2. $2a$
3. 4
4. 3
5. 8
6. a
7. 2
8. $3b$
9. $\frac{7a}{2}$
10. $\frac{x^2}{3}$

Exercise 1M

1. a^3
2. $2mn$
3. hk^2
4. d^2ef
5. $8b$
6. $9b^2$
7. 0
8. 2
9. 1
10. x
11. $\frac{1}{x}$
12. $8m$
13. $7d$
14. $\frac{2f}{3}$
15. $3a$
16. $\frac{5b}{2}$
17. 1
18. $\frac{1}{a}$
19. $\frac{h}{g}$
20. b^2
21. ab^3
22. $\frac{2}{m^2}$

Exercise 2A

1. $A = \frac{3N}{100}$
2. $W = 18x$
3. $V = \frac{z}{8}$
4. $a = 6c$
5. $b = \frac{nc}{100}$
6. $d = 12b$
7. $t = \frac{4}{v}$
8. $x = \frac{370}{y}$
9. $p = xy$
10. $t = \frac{300}{v}$

Exercise 2B

1. $2(x + y)$
2. $3(2n + m)$
3. $x(a - 4)$
4. $4(s + t)$
5. $a(n - 1)$
6. $2(25 + 2x)$ p

Exercise 2C

1. $(a + 3) \div 4$, or $\frac{a+3}{4}$, or $\frac{1}{4}(a + 3)$
2. $\frac{1}{2}(n + m)$

3. $\frac{1}{5}(a-b)$
4. $\dfrac{x+y}{1000}$
5. $\dfrac{a+c+2d}{20}$
6. $1000\,(2a+40)$
7. $\dfrac{8-t}{36}$
8. $x\,(x+3)$
9. $\dfrac{12-N}{5}$
10. $\dfrac{y-x}{3}$

Exercise 2D

1. $2x+2y$
2. $20a+5b$
3. $3x+xy$
4. $6d+3e+6f$
5. $ab-ac$
6. $2ax+4ay$
7. $48d+8df$
8. $35y+7z+63$
9. $3x^2+3x$
10. a^2+a^3
11. 3
12. 10
13. 5
14. 144
15. 10
16. 25
17. 13
18. $1\frac{1}{3}$
19. 8
20. 18

Exercise 2E

1. 130 yd; $p=2\,(x+y)$
2. 95°; $x=180-y$
3. $b=180-3a$
4. $A=3s+2t$
5. 45 in³; $C=a^2b$
6. £270; $x=120+30y$
7. £19; $A=7n+3p$
8. $m=\dfrac{5x}{3}$
9. (i) $a=\dfrac{11-b}{2}$; or $a=\frac{1}{2}(11-b)$
   (ii) $b=11-2a$
10. $(100+48x)$ gal; $(100+mx)$ gal; $G=100+hx$
11. 10p; $2T+2$
12. 12p; $3x$ pence; $(2b+3c)$ pence; $\dfrac{(2b+3c)}{(b+c)}$ pence; $C=\dfrac{2b+3c}{b+c}$
13. $1+4\,(n-1)$
14. $y=\dfrac{x}{1000}$
15. 63 ft; $C=DE$
16. $180-y$
17. $(b+c)$ knots; $(b-c)$ knots;

$\left(\dfrac{d}{b+c}\right)$ h; $\left(\dfrac{d}{b-c}\right)$ h;

$T=\dfrac{d}{b+c}+\dfrac{d}{b-c}$

Exercise 2F

1. 21·6 hp
2. 20 yr; 2 yr
3. 500 kg
4. $1\frac{4}{7}$ seconds
5. 40 miles
6. 1 cwt; No
7. No, less than $2\frac{2}{5}$ tons
8. £60
9. 50 runs
10. 11 tons

Exercise 3A

1. 5
2. 6
3. 8
4. 0
5. $1\frac{1}{4}$
6. 30
7. 6
8. 9
9. 3
10. 1
11. 6
12. 7
13. 2
14. 4
15. 4

Exercise 3B

(x has been used to stand for the unknown; any letter will do)

1. $6 + 4x = 18$
2. $2x + 7 = 13$
3. $\frac{x}{4} + 8 = 12$
4. $x + 15 = 4x$
5. $4x + 2x = 24$
6. $x - 20 = \frac{x}{3}$
7. $x \times 2x = 18$
8. $150 - x = 126$
9. $180 - 40 - 50 = x$
10. $w + \frac{w}{4} = 20$

Exercise 3C

1. 12
2. 17
3. 929
4. 42
5. 14
6. 16
7. 0
8. 3
9. 11
10. 1
11. $\frac{3}{8}$
12. $2\frac{1}{5}$
13. 0
14. 5
15. 63
16. 0
17. $1\frac{1}{4}$
18. $\frac{1}{2}$
19. $\frac{3}{4}$
20. 5

Exercise 3D

1. 24
2. 11
3. $6\frac{1}{4}$
4. $\frac{7}{11}$
5. $1\frac{1}{2}$
6. 1
7. 6
8. 9
9. 6
10. 7
11. 5
12. 4
13. 5
14. 9
15. 5
16. 25
17. 5
18. 2
19. 4
20. 6

Exercise 3E

1. 0
2. 3
3. 20p
4. $x + 1$; $x - 1$; 46
5. $x + 2$; $x - 2$; 32
6. $2\frac{5}{7}$
7. 56 cars
8. 45p
9. 30; right-angle

10. 21 ft.
11. 4p
12. 49
13. 3 weeks
14. 8 years old
15. $x°$; $(180 - x)°$; 45°
16. 44
17. 30
18. Paul 87 marks, Luke 100
19. 30°C
20. 3 in

Exercise 3F

1. False
2. False
3. False
4. True
5. True
6. True
7. True
8. True
9. False
10. True

Exercise 3G

1. Any number less than 2
2. Any number greater than 6
3. Any number greater than 4
4. Any number greater than 14
5. Any number greater than 4
6. Any number less than 4
7. Any number greater than 10
8. Any value except 3
9. Any value except 4
10. Any value except 36

Exercise 4A

1. m^4
2. a^5
3. $6b^3$
4. c
5. e^5
6. $2f$
7. $9x^2$
8. $8x^6$
9. x^3
10. $3y^2$
11. g^7
12. h^{10}
13. $5k^4$
14. $3l^5$
15. $125m^9$
16. $9p^6$
17. $2a^2b$
18. $2a^3b$
19. $5r^4s^2$
20. t
21. u
22. 1
23. w^4
24. $2y$
25. 4
26. $\frac{1}{4}$
27. $\frac{a}{5}$
28. $\frac{3}{a}$
29. 26
30. $\frac{1}{c}$
31. a^6
32. $27d^6$
33. $9d^6$
34. $16e^4$
35. 4
36. $2x$
37. $6y^3$
38. $3x^2$
39. ab
40. cd^2
41. e^3f^4
42. $81g^2h^4$
43. $\frac{3}{k1}$
44. $\frac{4a^2}{3}$
45. k^3
46. $\frac{2}{p}$
47. $\frac{1}{4}x^2y$, or $\frac{x^2y}{4}$
48. x
49. $2a$
50. $2m$

Exercise 4B

1. $2x$
2. $2a$
3. $3b$
4. c^2

5. 1 7. $3xy^2$ 9. $5a^2b^2c^2$
6. $4h$ 8. $2r^2st$ 10. $11xy^3$

Exercise 4C

1. $2^4 \times 3^2 \times 5^2$ 4. $8m^2n^2$ 7. $12a^2b^2$ 10. $4x^2y^2$
2. $6x^2$ 5. $125rs$ 8. $35x^2y^4$
3. $20xy^3$ 6. $18x^3y^4$ 9. $60x^4$

In the following answers the H.C.F. is given first.

11. a, a^3 13. $bcd, 12b^2c^2d$ 15. $1, 91a^2bgx^2$ 16. $2, 24a^2b^3$
12. $11y, 121x^2y$ 14. $3ef, 27e^2f^5$

Exercise 4D

1. $\frac{x}{y}$
2. $\frac{a}{2b}$
3. $\frac{3x}{4z}$
4. 1
5. $\frac{7}{5c}$
6. $3r$
7. s
8. t^6
9. $\frac{3}{5}$
10. $4a^2$
11. $3b^2$
12. $\frac{11c}{2}$
13. cf
14. $\frac{6x^2}{5}$
15. $\frac{1}{3xy}$
16. $\frac{ef}{4}$
17. $\frac{1}{k^6}$
18. $\frac{4b^4}{3a}$
19. y
20. $\frac{1}{2}$
21. $\frac{x^3}{5}$
22. x^4
23. xy
24. $\frac{3ac^2}{b}$
25. $\frac{1}{3x^3y^3}$
26. $\frac{5nx^3}{2}$
27. $\frac{7}{8ab}$
28. $2x^2y^3$
29. $\frac{a^2}{3y^2}$
30. $\frac{x}{4y}$

Exercise 4E

1. b^9
2. $2a^6$
3. $\frac{a^3}{b^6}$
4. $\frac{2a^{11}}{15b^6}$
5. $\frac{4ab^2}{15c^2d}$
6. $\frac{2x^4}{y^4}$
7. $\frac{35y}{2x}$
8. xy
9. $\frac{a^3}{x}$
10. $\frac{y}{2b}$

Exercise 4F

1. $\frac{7}{27}$
2. 7
3. $\frac{a}{bc}$
4. y
5. $\frac{2a^2}{b}$
6. $\frac{l}{pm}$
7. 1
8. $\frac{4f^3}{9}$
9. $\frac{4y^2}{x}$
10. $\frac{6}{b}$

Exercise 4G

1. 3, 11
2. 15, 45
3. $2x$
4. ac^2
5. ef^2
6. $2x^2$
7. $2xy$
8. $4an$
9. $6ax^3y$
10. $10mp^2$

Exercise 4H

1. $\frac{7a}{10}$
2. $\frac{10a}{21}$
3. $\frac{5x}{9}$
4. $\frac{23x}{20}$
5. $\frac{5}{x}$
6. $\frac{13}{2x}$
7. $\frac{19}{6x}$
8. $\frac{10}{3y}$
9. $\frac{1}{2x}$
10. $\frac{3b}{70}$
11. $\frac{3x-2}{54}$
12. $\frac{4x-5}{20}$
13. $\frac{7}{x}$
14. $\frac{7m}{12}$
15. $\frac{31s}{30}$
16. $\frac{7}{3a^2}$
17. $\frac{5x+3}{x^2}$
18. $\frac{ac-b}{c}$
19. $\frac{31}{6x}$
20. $\frac{1}{a}$

Exercise 4J

1. $\frac{b-a}{ab}$
2. $\frac{2y+x}{2xy}$
3. $\frac{3p-4m}{2mp}$
4. $\frac{15b+2a}{6ab}$
5. $\frac{4v+3w}{28}$
6. $\frac{25a-2b}{25}$
7. $\frac{6e+ef}{4}$
8. $\frac{5a-4}{2a}$
9. $\frac{ad-bc}{bd}$
10. $\frac{bx+by+ay}{by}$

Exercise 4K

1. $\frac{5}{24}, \frac{5n}{24}$
2. $\frac{1}{8}, \frac{x}{8}$
3. $p-n; \frac{n}{p}; \frac{p-n}{p}$
4. $\frac{7y}{3}$ yd^2
5. $\frac{15}{n}$p
6. $\frac{10n}{3}$ p; $\frac{10}{3}(n+1)$ p
7. $\frac{3x}{4}$ gal; $\left(x-\frac{y}{z}\right)$ gal
8. $\frac{8n}{5}$ km, $\frac{32n}{5}$ km
9. $x-n; \frac{n}{x}; \frac{x-n}{x}$
10. $\frac{y-x}{y}$, or $1-\frac{x}{y}$

11. $\frac{6}{35}$; $1 - \frac{5x}{6y}$; or $\frac{6y - 5x}{6y}$

12. $\frac{5x}{n}$ hr.; $\frac{300x}{n}$ min.

13. $\frac{2n^2}{15}$ ft^2; $\frac{26n}{15}$ ft

14. $\frac{a^3}{2}$ in^3; $\frac{23a^2}{6}$ in^2

15. n metres

Exercise 5A

1. $P = \frac{4W}{5}$
2. $n = \frac{r + 4}{2}$; 4
3. $[180 - (b + c]^b$
4. $10\frac{1}{2}$
5. $\frac{P}{2} - L = B$
6. $h = \frac{2A}{b}$
7. $x = \frac{22V}{15}$; 66 ft/s
8. $R = \frac{100I}{PT}$; 4%
9. $r = \frac{c}{2\pi}$
10. $y = \frac{x}{1 + a}$

Exercise 5B

1. $r = \frac{\sqrt{A}}{\pi}$
2. $C = \frac{5}{9}(F - 32)$
3. $\frac{2}{3}$ ft
4. $a = \frac{2A}{h} - b$; 3 in
5. $x = \sqrt{t} + 1$
6. $d = \sqrt{\frac{L}{8}}$; 2 in
7. $R = \sqrt{\left(\frac{A}{\pi} + r^2\right)}$; 15 ft
8. $f = \frac{2(s - ut)}{t^2}$
9. $t = \frac{2s}{u + v}$
10. (i) $b = \frac{4lW}{5d^2}$

    (ii) $d = \sqrt{\left(\frac{4lW}{5b}\right)}$

    (iii) $l = \frac{5bd^2}{4W}$

Exercise 5C

1. 9 cows, 28 turkeys
2. 90 m, 122 m
3. 24
4. 11
5. $2\frac{1}{2}$ h
6. £1·50; 90p
7. 12 sums
8. 1 in; 28 in
9. 10; less than 10
10. 6 lb; 24 lb
11. 4
12. 192 km
13. 13, 15, 17, 19
14. 550 mile/h; 50 mile/h
15. 30 in
16. $1\frac{3}{4}$ h
17. 7×1p, 4×5p, 2×10p
18. 1 cm
19. 50 men, 106 women
20. 6 min

Exercise 5D

1. 5 doz.; 2 doz.
2. -18
3. £12
4. 16 marks
5. Fish £55, Chips £50
6. 79
7. 10.16 a.m.
8. 240 mile/h
9. 12 years old
10. 3 in

Exercise 6A

1. $+20°C$; $-35°C$; $-2\frac{1}{4}°$ C
2. £$(+50)$, £(-10)
3. (a) ...5 m *below* sea-level
   (b) ...*lost* 5 lb in weight
   (c) ... 2 years *younger* than I
   (d) ... is 12 minutes *slow*
   (e) ... started 25 years *before* ...
   (f) ... is 3 miles *south* of mine
4. £$(+50)$ £(-56)
   £$(+10)$ £(-185)
   £(-25) £$(+30)$
5. 2.05 p.m.; 2.18 p.m.; 2.28 p.m.; 2.30 p.m.; 2.50 p.m.
6. $-2, +6, +15, -12, 0, -14$
7. $+1, -2, 0, -3, 19$
8. On just before 11 p.m., off just after 9 a.m.

Exercise 6B

1. $+12$
2. $+4$
3. -1
4. $+14{\cdot}5$
5. $+2$
6. -13
7. -5
8. -6
9. -17
10. -3
11. $+8$
12. -8
13. -1
14. -2
15. -3
16. -4
17. $-1, +5, -2, 8$
18. $-a, -5a, -4a, 2a$
19. $y-x$; $-3y$; $-3b$; $2a$; $-2a$
20. $-6+a$; $-2y$; $-2x$; $+2n$; $-4a$

Exercise 6C

1. -12
2. -12
3. $+63$
4. -24
5. -51
6. -27
7. -60
8. -18
9. -32
10. $+4$
11. $+16$
12. $+9$
13. -8
14. $+\frac{1}{4}$
15. $-2x^2$
16. $+3x^2$
17. $-x^2$
18. $-xy^2z$
19. $+18x^2$
20. $+12ab$
21. $+24$
22. -18
23. $+12$
24. -10
25. -5
26. -2
27. -14
28. $+28$
29. -12
30. $+5$

Exercise 6D

1. -3
2. $+6$
3. -2
4. -9
5. $+3$
6. -3
7. $+6$
8. -4
9. -1
10. -1
11. $+1$
12. -1
13. $-\frac{1}{2}$
14. -2
15. $+4x$
16. $-y$
17. -2
18. $-2a$
19. $+3x$
20. $\frac{-xy}{z}$
21. $3-x$
22. $3-x$
23. $-x-3$
24. -20
25. -80
26. -10
27. -40
28. $+29$
29. $13\frac{1}{2}$
30. -25
31. 0
32. 0
33. $-2\frac{2}{5}$
34. $-2\frac{1}{2}$
35. $-1\frac{3}{5}$

Exercise 6E

1. $+1$
2. -1
3. $+13$
4. $+1$
5. $+6$
6. $-3x$
7. $+y$
8. $-10n$
9. $+21ab$
10. $-4x-8y$
11. $2x+w$
12. $-w$
13. $5a-b$
14. $-x$
15. $3a^2-6a+6$
16. $5n-6$
17. $4n+1$
18. $x+1$
19. $7a$
20. $2b$
21. $2a$
22. $2x+y$
23. $e+6f$
24. $3e-6f$
25. $-6x-5y$
26. 0
27. $9x$
28. $m+n-1$
29. $-2x^2-6x-4$
30. $a+b$
31. $5x+8y+3z$
32. $4m+3n$

Exercise 6F

1. $-(-x+y)$
2. $x+(2y-3z)$
3. $x-(-y+2)$
4. $3x-(y-6)$
5. $-(-x^2+5x-6)$
6. $3(m+7n)$
7. $5(2a-5b)$
8. $2e(2-f)$
9. $x^2(5x-2)$
10. $-a(-2-b)$

Exercise 6G

1. -5
2. -4
3. -2
4. $+2$
5. $+2\frac{1}{2}$
6. -3
7. -4
8. -5
9. -3
10. $+4$
11. $+2$
12. -3
13. $+4$
14. $+11$
15. $+3$
16. -4
17. $+2$
18. -1
19. -9
20. $+5$

Exercise 7A

1. Meat
2. 3p
3. 1p; 12p

Exercise 7D

1. Probably the 6th. His marks improved
2. The 20-cm measurement; 18·7 cm; 21 cm; 3·3 kg

Exercise 7E

1. 35 yd. 43 mile/h
2. $2\frac{1}{2}$ h; $11\frac{1}{4}$ miles; $19\frac{1}{2}$ miles; 10.07 a.m.
3. 2176 ft; 2400 ft; No; 2500 ft approx.; 7 s; 18 s

Exercise 8A

1. 50
2. 187
3. 30
4. 54
5. 24
6. 8
7. 56
8. 170
9. 230
10. 26
11. $17a$
12. $5mx$
13. $2a(3y + 2b)$
14. $5x(2y + 3z)$

Exercise 8B

1. $2a + ab + b + 2$
2. $4x + xy + 3y + 12$
3. $5m + mn + 3n + 15$
4. $8c + cd + 12d + 96$
5. $3g + fg - 2f - 6$
6. $2f - fg - 3g + 6$
7. $xy - 2x - 7y + 14$
8. $20x^2 + 67x + 39$
9. $x^2 + 13x + 40$
10. $a^2 + \frac{13a}{2} + 10$
11. $a^2 - 49$
12. $25 - x^2$
13. $x^2 + 8x + 16$
14. $a^2 - 6a + 9$
15. $16p^2 - 16p + 4$
16. $9x^2 + 42x + 49$
17. $14x^2 + 9xy + y^2$
18. $x^2 + 2xy + y^2$
19. $x^2 - 2xy + y^2$
20. $x^2 - y^2$

Exercise 8C

1. 1 or -5
2. 9 or -1
3. -1 or -13
4. 3 or -1
5. 10 or -8
6. $\frac{2}{3}$ or -2
7. 78
8. 9
9. 7
10. -1 or $-6\frac{1}{3}$

Exercise 8D

1. $\dfrac{5a + 7}{6}$
2. $\dfrac{8b + 50}{15}$
3. $\dfrac{13x + 4}{42}$
4. $\dfrac{4x + 27}{10}$

5. $\dfrac{5a + 14}{12}$
6. $\dfrac{-37a - 2}{72}$
7. $\dfrac{7n - 17}{8}$
8. $\dfrac{21 - 11m}{12}$
9. $\dfrac{3x + 16}{4}$
10. $\dfrac{22x - 31}{15}$
11. $\dfrac{72 - 2x}{15}$
12. $\dfrac{-14b - 106}{15}$
13. $\dfrac{19b - 8a}{35}$
14. $\dfrac{47(e + f)}{45}$
15. $\dfrac{4a + 2b}{3c}$
16. $\dfrac{3x - 8v}{3z}$
17. $\dfrac{4a - b + c}{c}$
18. $\dfrac{2x - 12v}{35n}$
19. $\dfrac{2c^2 + 23cd - 5d^2}{10cd}$
20. $\dfrac{3x^2 + xy + 4x - 14y}{2xy}$
21. $\dfrac{8g^2h + 7gh^2 - 2gn + 2hn}{ghn}$
22. $\dfrac{2a^2 + a^2b + 17b}{b^2}$

Exercise 8E

1. 5
2. 4
3. $\dfrac{1}{12}$
4. $\dfrac{10}{17}$
5. 1
6. 6
7. $\dfrac{5}{29}$
8. $1\frac{1}{5}$
9. 18
10. $1\frac{11}{13}$
11. $-\frac{1}{4}$
12. $2\frac{2}{5}$
13. 3·5
14. 45
15. £12
16. $1\frac{1}{3}$
17. 2
18. $1\frac{7}{17}$
19. $\dfrac{74}{103}$
20. 1
21. $2\frac{2}{3}$
22. $2\frac{3}{5}$
23. $-1\frac{30}{79}$
24. $10\frac{1}{2}$
25. $-\frac{1}{64}$
26. $16\frac{5}{8}$

Exercise 8F

1. $6m + 1$
2. $24y + 30$
3. $35a + 42$
4. $2a^2 - 2b^2$
5. $6 + 18y + gy^2$
6. $5b^2 + 16b$
7. $3c - 6$
8. $2e + 3f$
9. $n^3 + 3n^2 + 3n$
10. $3y^2 + xy + 3x - 2x^2$
11. $\dfrac{15x}{2} + 13y$
12. $b - c - 5a$
13. $1 + 2x^3$
14. $2a - 2b$
15. 0
16. $6\frac{1}{2}$
17. $\dfrac{2}{21}$
18. 33
19. $+110$
20. 12
21. 4
22. 126

23. 23

24. -5

25. $4x + 6$

26. $3n - 3$

27. 38; $x - y$; $14a - 7b - 5x$

28. $\dfrac{14x + 8y}{3}$ yd

29. 5

30. (i) $\frac{1}{2}(c + d)$
 (ii) $\frac{1}{2}[\frac{1}{2}(a + b) + d]$

Exercise 9A

10 ft   5·3 s

Exercise 9B

1. After 3 h 40 min; 1 h 16 min late
2. 7 h 50 min

Exercise 9C

1. 12.20 p.m. $3\frac{1}{3}$ miles from A

Exercise 10A

1. $x = 2, y = 7$
2. $x = 2, y = 1$

Exercise 10B

1. $y = 3x + 4$
2. $v = 2t + 12$

Exercise 10C

1. $x = 4, y = 1$
2. $x = 3, y = 1$
3. $x = 2, y = -3$
4. $x = \frac{1}{2}, y = 1$
5. $x = -3\frac{4}{17}, y = -\frac{3}{17}$

Exercise 10D

1. $x = 5, y = 7$
2. $x = 2, y = 2$

Exercise 10E

1. $x = 2, y = -4$
2. $x = \frac{1}{2}, y = -2$
3. $x = 3, y = 3$

Exercise 10F

1. $y = 4x - 10$
2. $y = x + 6$
3. $y = -2x$

Exercise 10G

1. $x = 2, y = 5$
2. $x = -2, y = 3$
3. $x = 1, y = 2$
4. $x = -\frac{1}{2}, y = 2$
5. $x = 3, y = -1$
6. $x = 3, y = 4$
7. $x = -1, y = 2$

Exercise 10H

1. $a = 27, b = 19$
2. $f = 44, s = 12$
3. $p = 150, q = 58$

Exercise 10J

1. £3·25; 45p
2. 33, 11
3. 3 miles; 5 miles
4. 9 × 2p; 5 × 5p
5. 47
6. £1·70; 70p
7. $2\frac{1}{2}$ h; $\frac{1}{4}$ h r.

Exercise 11A

1. $2(x - 2y)$
2. $3b(a + 3c)$
3. $p(p - q)$
4. $x^2(x - y)$
5. $3(b + 1)$
6. $d(d - 1)$
7. $h(3h + 1)$
8. $s(2r - 1)$
9. $t(1 + 4t^2)$
10. $5ab(a - 2b)$
11. $4c(c^2 + 2)$
12. $2x(2y - z + 4x)$

Exercise 11B

1. $p^2 + q^2 + 2pq$
2. $x^2 + y^2 - 2xy$
3. $4a^2 + b^2 + 4ab$
4. $4c^2 + 25 - 20c$
5. $36x^2 + 1 + 12x$
6. $16y^2 + 49z^2 - 56yz$
7. $h^4 + k^4 + 2h^2k^2$

Exercise 11C

1. 40
2. 1260
3. 52 689
4. 20·8
5. 247

Exercise 11D

1. $a^2 - 25$
2. $9x^2 - 4$
3. $4p^2 - 81q^2$
4. $(y + 6)(y - 6)$
5. $(z + 1)(z - 1)$
6. $(2k + 3)(2k - 3)$
7. $(4x + 11y)(4x - 11y)$

Exercise 11E

1. $3(a+3)(a-3)$
2. $5(x+2y)(x-2y)$
3. $a(b+c)(b-c)$
4. $p(2x+3y)(2x-3y)$
5. $h(y+1)(y-1)$

Exercise 11F

1. $3hx+hy+6kx+2ky$
2. $6af-12ag+12bf-24bg$
3. $2cy+3cz-14dy-21dz$
4. $10as-15at-4bs+6bt$

Exercise 11G

1. $(a+3)(x+y)$
2. $(p+2q)(c-2d)$
3. $(x-y)(x+4)$
4. $(h-3k)(s-5t)$

Exercise 11H

1. $(a-b)(r-s)$
2. $(x-y)(x-1)$
3. $(a-b)(c+1)$
4. $(p-q)(p+q-x)$
5. $(r-1)(r+1+s)$

Exercise 11J

1. $2a^3+3a^2-23a+21$
2. $3p^3+7p^2q-8pq^2-6q^3$
3. x^3-4x^2+x+6

Exercise 11K

1. $(x^2+1)(x+1)(x-1)$
2. $(4p^2+9q^2)(2p+3q)(2p-3q)$
3. $(b+4)(a+1)(a-1)$

GEOMETRY

Exercise 1A

B. 2 steps at 12 o'clock $\dfrac{OB}{OR} = 2$ ORB is a straight line

C. 3 steps at 3 o'clock $\dfrac{OC}{OR} = 3$ $R\hat{O}C = 90°$

D. 1 step at 2 o'clock $\dfrac{OD}{OR} = 1$ $R\hat{O}D = 60°$

E. 2 steps at 4 o'clock $\dfrac{OE}{OR} = 2$ $R\hat{O}E = 120°$

H. 3 steps at 9.30 o'clock $\dfrac{OH}{OR} = 3$ $R\hat{O}H = 295°$

I. $2\frac{1}{2}$ steps at 11 o'clock $\dfrac{OI}{OR} = 2{\cdot}5$ $R\hat{O}I = 330°$

J. 3 steps at 1 o'clock $\dfrac{OJ}{OR} = 3$ $R\hat{O}J = 30°$

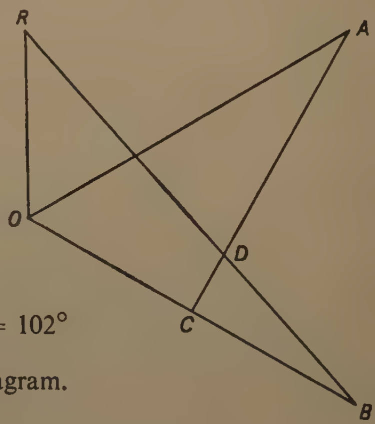
Exercise 1B

For answers **1-6** see diagram on facing page.

7. $\dfrac{OX}{OR} = 1{\cdot}65;\ R\hat{O}X = 150°$

8. $\dfrac{OY}{OR} = 1{\cdot}2;\ R\hat{O}Y = 90°$

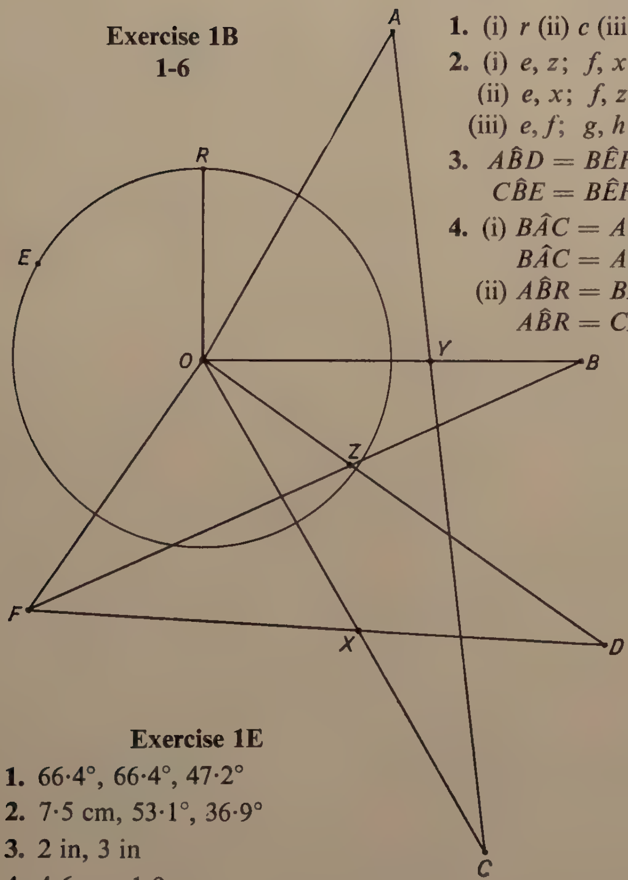
9. $\dfrac{OZ}{OR} = 0{\cdot}95;\ R\hat{O}Z = 125°$



10. $\dfrac{OD}{OR} = 1{\cdot}05\text{(approx.)};\ R\hat{O}D = 102°$

as shown in this diagram.

Exercise 1B
1-6



Exercise 1D

1. (i) r (ii) c (iii) q (iv) d
2. (i) $e, z;\ f, x;\ g, y;\ h, w$
   (ii) $e, x;\ f, z;\ g, w;\ h, y$
   (iii) $e, f;\ g, h;\ x, z;\ w, y$
3. $A\hat{B}D = B\hat{E}F$ (cor.)
   $C\hat{B}E = B\hat{E}F$ (alt.)
4. (i) $B\hat{A}C = A\hat{B}R$ ($AC \parallel BR$; alt.)
   $B\hat{A}C = A\hat{C}Q$ ($AB \parallel CQ$; alt.)
   (ii) $A\hat{B}R = B\hat{A}C$ ($BR \parallel AC$; alt.)
   $A\hat{B}R = C\hat{P}B$ ($AB \parallel CP$; cor.)
   (iii) $\hat{P} = A\hat{B}R$
   ($CP \parallel AB$; cor.)
   $\hat{P} = A\hat{C}Q$
   ($PB \parallel CA$; cor.)

Exercise 1E

1. 66·4°, 66·4°, 47·2°
2. 7·5 cm, 53·1°, 36·9°
3. 2 in, 3 in
4. 4·6 cm, 1·9 cm
5. 4·8 cm
6. 073°

Exercise 2A

1. $\dfrac{PQ}{AB} = \dfrac{GH}{AB} \cdot \dfrac{PQ}{GH}$
2. $\dfrac{HK}{RS} = \dfrac{HK}{LM} \cdot \dfrac{LM}{RS}$
3. $\dfrac{AB}{BC} \cdot \dfrac{BC}{CD} = \dfrac{AB}{CD}$
4. $\dfrac{WX}{YZ} = \dfrac{WX}{XY} \cdot \dfrac{XY}{YZ}$
5. $\dfrac{PQ}{AB} = 6$
6. $\dfrac{2}{5}$
7. xy
8. $\dfrac{p}{q}$
9. 4·44
10. (i) 6. (ii) 9·6
11. 10

Exercise 2B

1. (a) $17\frac{1}{2}$
   (b) $17{\cdot}9881$
   (c) $5{\cdot}77$
   (d) 1
   (e) $\frac{3}{4}$
2. (a) $AB = 5$
   (b) $AB = \pm 6$
   (c) $XY = 15$
   (d) $GH = 12$
3. (a) $AB(XY + XZ)$
   (b) $PQ(PQ - PR)$
   (c) $(XY + XZ)(AB - CD)$
   (d) $AB(AB - 1)$
4. $AB = 6$
5. $PT = 21,\ ST = \dfrac{49}{3}$
6. $AD = 3,\ BD = 2\frac{1}{2}$
7. $DE = 3{\cdot}9$

Exercise 2C

2. $DE = 3$
3. $KN = 12$
4. 17 miles
5. 10 m

Exercise 2D

1. $9{\cdot}434$ m
2. $TV = 21$
3. $\sqrt{3}$ or $1{\cdot}732$ approx.
4. $OD = 2$
6. 13 m
7. (i) 5 miles
   (ii) 12 miles
   (iii) 13 miles
8. 28 cm
9. $\sqrt{8}$ or $2{\cdot}828$ approx.
10. $5{\cdot}77$ ft

Exercise 3A

1. $\dfrac{25}{3}$
2. $\dfrac{121}{4}$
3. $8{\cdot}478$
4. 32

Exercise 3B

1. $AXBY = ABX + ABY$
2. $AXBY = AXY - BXY$
3. $PQR = PQS + PRS$
4. $DEFG$
5. $DEFG$
7. $KMN = KLN - KLM$
8. $RSVUT$
9. $STUV$

Exercise 3D

1. $42{\cdot}5$ cm²
2. $9{\cdot}45$ cm
3. 5 in, 3 in
4. 8 in²
5. $ORA = 20$ in²
   $ORC = 14$ in²
   $ARC = 2$ in²
6. $AM = 7{\cdot}1$ cm approx.
   $ABC = 17{\cdot}7$ cm² approx.
7. 60 cm²

Exercise 3E

1. (a) $\frac{3}{8}$
   (b) $\frac{8}{5}$
   (c) $\frac{3}{5}$
   (d) $\frac{5}{8}$
   (e) $\frac{3}{2}$
   (f) $\frac{15}{16}$
   (g) $\frac{9}{16}$
2. (a) $\dfrac{PS}{SQ}$
   (b) $\dfrac{PS}{PQ}$
   (c) $\dfrac{PR}{PT}$
   (d) $\dfrac{PS}{SQ}$
   (e) $\dfrac{PT}{PR}$
4. (a) $\dfrac{PS}{SQ} = \dfrac{PSR}{SQR}$
   (b) $\dfrac{PTQ}{PRQ}$
   (c) $\dfrac{PSR}{PQR}$
7. $\dfrac{ABC}{ADE} = 4$

Exercise 4A

1. 149°
2. 142°
3. 162°
4. $K\hat{L}M = 150°$, $K\hat{M}L = 15°$
5. $n = 15$

Exercise 6A

7. 60°

Exercise 6D

7. $MC = 35$, $LC = 56$
8. $BD = 4$

Exercise 7A

4. (i) $\sqrt{119}$
   (ii) 10·91
5. $AB = 9$; $OAB = 36$
7. $\sqrt{288}$ or $12\sqrt{2}$; 16·97
9. 59·40 cm²
10. 13

Exercise 7B

1. (6, 8); (8, 4)
2. 5, 2; 7, − 2
3. 5, 20, 25
6. 5, 12, 13
7. 3, 4, 5
8. $(x-4)^2 + (y-7)^2$
10. (a) 0·8, 1·5, 1·7
    (b) 3·5, 1·2, 3·7
    (c) $2\frac{1}{4}$, $3\frac{1}{2}$, $\dfrac{\sqrt{277}}{4}$ or 4·16 approx.

Exercise 8A

1. *AP* comes out of the circle at *Z*
   AQ comes out of the circle at *Y*
   AR comes out of the circle at *X*
   AS comes out of the circle at *W*
   AT comes out of the circle at *A*

2. (a) Z is a point on the minor arc AY
   (b) W (or X or Y) is a point on the major arc AZ
   (c) AWZ (or AXZ or AYZ) is an angle in the segment AYZ
   (d) AWY (or AXY) is an angle in the segment AWY)
3. $T\hat{A}W = A\hat{X}W$ (or $A\hat{Y}W$ or $AW\hat{Z}$)
4. $A\hat{X}W = A\hat{Y}W = A\hat{Z}W$
5. $A\hat{W}X = 1$ rt.-angle; $A\hat{Y}X = 1$ rt.-angle; $A\hat{Z}X = 1$ rt.-angle
6. $CP = CA = CQ$. CPA is isosceles, having $CP = CA$; CQA is isosceles, having $CA = CQ$. $C\hat{P}A = C\hat{A}P$; $CQA = C\hat{A}Q$
7. $P\hat{C}K = C\hat{A}P + C\hat{P}A$ (ext. angle of a triangle CAP)
   $= 2.C\hat{A}P$ ($C\hat{P}A = C\hat{A}P$)
   $= 2 \times 23° = 46°$
   $Q\hat{C}K = C\hat{A}Q + C\hat{Q}A$ (ext. angle)
   $= 2.C\hat{A}Q$ ($C\hat{Q}A = C\hat{A}Q$)
   $= 2 \times 32° = 64°$
   $P\hat{C}Q = P\hat{C}K + Q\hat{C}K$
   $= 46° + 64° = 110°$

   Note: The argument is here repeated with Q for P and 32° for 23°, so it would be sufficient to say: Similarly $Q\hat{C}K = 2 \times 32°$
   $= 64°$
8. XBY is an angle in the minor segment XY of the circle centre N. XCY is an angle in the major segment XY of the circle centre N.
9. XA is a tangent to the circle centre N; XB is a tangent to the circle centre M.
   XA enters the circle centre M at X and leaves at A
   XA enters the circle centre N at X and leaves at X
   XB enters the circle centre M at X and leaves at X
   XB enters the circle centre N at X and leaves at B
10. The third line is XCD, entering the circle centre M at X and leaving at D; it enters the circle centre N at X and leaves at C.
    The line YCA enters both circles at Y; it leaves the circle centre M at A, and it leaves the circle centre N at C. The line BYE enters both circles at Y and leaves the circle centre M at E; it leaves the circle centre N at B.

Exercise 8C

6. In $\triangle ABD$, $A = 80°$ $B = 50°$ $D = 50°$
   In $\triangle CBD$, $C = 100°$ $B = 30°$ $D = 50°$

Exercise 8J

4. 5 cm   7. 1 in or 7 in   9. 2 cm

WEIGHTS AND MEASURES

Measures of Length

12 inches = 1 foot
3 feet = 1 yard
22 yards = 1 chain
10 chains = 1 furlong
8 furlongs = 1 mile
5280 feet = 1 mile
1760 yards = 1 mile
3 miles = 1 league

Nautical Measures

6 feet = 1 fathom
100 fathoms = 1 cable's length

One knot equals a speed of 1 British nautical mile per hour and is *not* a measure of distance.

Square and Land Measures

144 square inches = 1 square foot
9 square feet = 1 square yard
$30\frac{1}{4}$ square yards = 1 square rod, pole or perch
40 square rods = 1 rood
4 roods = 1 acre
640 acres = 1 square mile
4840 square yards = 1 acre

Cubic or Solid Measures

1728 cubic inches = 1 cubic foot
27 cubic feet = 1 cubic yard

Avoirdupois Weight

16 drams = 1 ounce
16 ounces = 1 pound
14 pounds = 1 stone
2 stones = 1 quarter
4 quarters = 1 hundredweight
20 hundredweights = 1 ton

In the United States, 100 pounds = 1 hundredweight. and the ton of 20 hundredweights = 2000 pounds. In order to distinguish between the two standards the terms "long" and "short" are used.

Long ton = 2240 pounds
Short ton = 2000 pounds
Metric ton (tonne) = 2204·6 pounds

Measures of Capacity

4 gills = 1 pint
2 pints = 1 quart
4 quarts = 1 gallon
2 gallons = 1 peck
4 pecks = 1 bushel
8 bushels = 1 quarter

Fractions of £1

Fraction	*Decimal*	*Value*
$\frac{1}{40}$	0·025	$2\frac{1}{2}$p
$\frac{1}{20}$	0·05	5p
$\frac{1}{10}$	0·1	10p
$\frac{1}{8}$	0·125	$12\frac{1}{2}$p
$\frac{1}{4}$	0·25	25p
$\frac{3}{8}$	0·375	$37\frac{1}{2}$p
$\frac{1}{2}$	0·5	50p
$\frac{5}{8}$	0·625	$62\frac{1}{2}$p
$\frac{3}{4}$	0·75	75p
$\frac{7}{8}$	0·875	$87\frac{1}{2}$p

Nautical Speeds in Miles per Hour

Knots	*Miles/h.*	*Knots*	*Miles/h.*
1 ..	1·1515	16 ..	18·4242
2 ..	2·3030	17 ..	19·5757
3 ..	3·4545	18 ..	20·7272
4 ..	4·6060	19 ..	21·8787
5 ..	5·7575	20 ..	23·0303
6 ..	6·9090	21 ..	24·1818
7 ..	8·0606	22 ..	25·3333
8 ..	9·2121	23 ..	26·4848
9 ..	10·3636	24 ..	27·6363
10 ..	11·5151	25 ..	28·7878
11 ..	12·6666	26 ..	29·9393
12 ..	13·8180	27 ..	31·0908
13 ..	14·9696	28 ..	32·2424
14 ..	16·1212	29 ..	33·3939
15 ..	17·2727	30 ..	34·5454

THE METRIC SYSTEM

Measures of Length

10 millimetres = 1 centimetre
10 centimetres = 1 decimetre
10 decimetres = 1 metre
10 metres = 1 decametre
10 decametres = 1 hectometre
10 hectometres = 1 kilometre

Measures of Capacity

10 millilitres = 1 centilitre
10 centilitres = 1 decilitre
10 decilitres = 1 litre
10 litres = 1 decalitre
10 decalitres = 1 hectolitre
10 hectolitres = 1 kilolitre
1 litre = 1 cubic decimetre
1 kilolitre = 1 cubic metre

Cubic Measures

1000 cubic millimetres = 1 cubic centimetre
1000 cubic centimetres = 1 cubic decimetre
1000 cubic decimetres = 1 cubic metre

Measures of Weight

10 milligrammes = 1 centigramme
10 centigrammes = 1 decigramme
10 decigrammes = 1 gramme
10 grammes = 1 decagramme
10 decagrammes = 1 hectogramme
10 hectogrammes = 1 kilogramme
1000 kilogrammes = 1 tonne
1 gramme = the weight of 1 cubic centimetre of water at 4° C (39° F).

1 kilogramme, or cubic decimetre, of water measures 1 litre.

Square Measures

100 square millimetres = 1 square centimetre
100 square centimetres = 1 square decimetre
100 square decimetres = 1 square metre
100 square metres = 1 square decametre
100 square decametres = 1 square hectometre or 1 hectare
100 square hectometres = 1 square kilometre

BRITISH AND METRIC EQUIVALENTS

British

1 inch = 25·4 millimetres
1 foot = 0·3048 metre
1 yard = 0·9144 metre
1 mile = 1·6093 kilometres

1 square inch = 6·452 square centimetres
1 square foot = 9·290 square decimetres
1 square yard = 0·836 square metre
1 acre = 0·405 hectare
1 square mile = 259 hectares

1 cubic inch = 16·387 cubic centimetres
1 cubic foot = 0·028 cubic metre
1 cubic yard = 0·765 cubic metre

1 ounce = 28·350 grammes
1 pound = 0·4536 kilogramme
1 hundredweight = 50·8 kilogrammes

1 gill = 1·42 decilitres
1 pint = 0·57 litre
1 quart = 1·136 litres
1 gallon = 4·546 litres

Metric

1 millimetre = 0·0394 inch
1 centimetre = 0·3937 inch
1 metre = 39·3708 inches
1 kilometre = 1093·63 yards

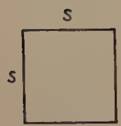
1 square centimetre = 0·155 square inch
1 square metre = 1·196 square yards
1 are = 119·6 square yards
1 hectare = 2·471 acres

1 cubic centimetre = 0·061 cubic inch
1 cubic metre = 1·308 cubic yards

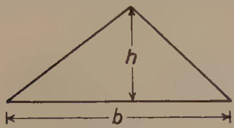
1 centigramme = 0·154 grain
1 gramme = 15·432 grains
1 kilogramme = 2·2046 lb
1 tonne = 0·984 ton

1 centilitre = 0·07 gill
1 litre = 1·761 pints
1 hectolitre = 2·75 bushels
1 hectolitre = 2·83 U.S. bushels
1 kilolitre = 3·44 quarters

USEFUL FORMULAE

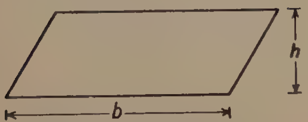


Square
Area $A = s^2$

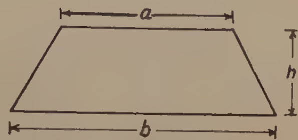


Triangle
Area $A = \frac{1}{2}b \times h$

Rectangle
Area $A = l \times b$

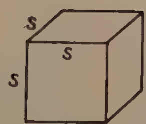


Parallelogram
Opp. sides are parallel
Area $A = b \times h$

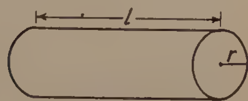


Trapezium
One pair of opp. sides parallel
Area $A = \frac{(a+b)}{2} \times h$

Triangular Prism
Volume $V = \frac{1}{2}b \times h \times l$

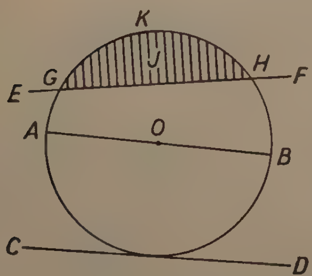


Cube
Volume $V = s^3$

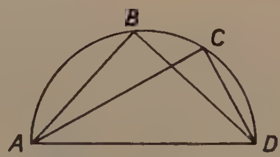


Cylinder
Volume $V = \pi \times r^2 \times l$

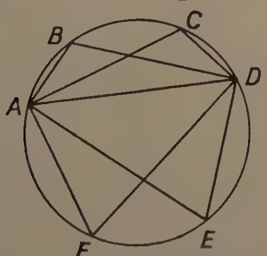
CIRCLES



Radius ______ AO
Tangent ______ CD
Diameter ______ AB
Chord ______ GH
Secant ______ EF
Arc (minor) ___ GKH
Segment (minor) __ J

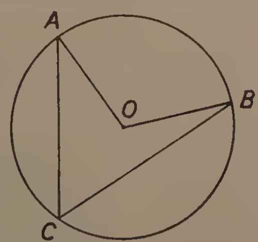


Angles in a semicircle are right-angles. $A\hat{B}D = A\hat{C}D = 90°$



Angles in the same segment are equal.
$A\hat{B}D = A\hat{C}D$
$A\hat{F}D = A\hat{E}D$

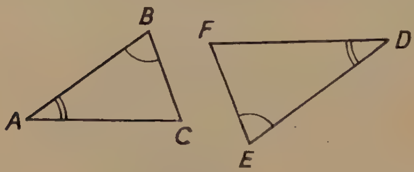
Angles in opposite segments are supplementary.
$A\hat{B}D + A\hat{F}D = 180°$
$A\hat{C}D + A\hat{E}D = 180°$



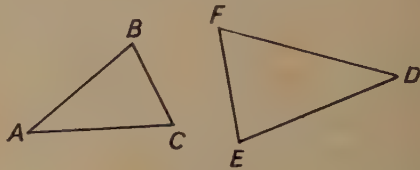
The angle subtended by an arc at the centre of a circle is double the angle it subtends at any other part of the circumference.
$A\hat{O}B = 2.A\hat{C}B$

SIMILAR TRIANGLES

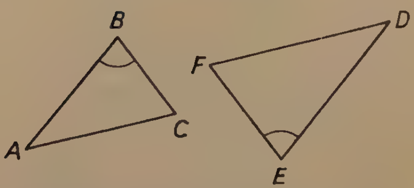
Two triangles are similar (i.e. agree about all shape-items) if they agree about:



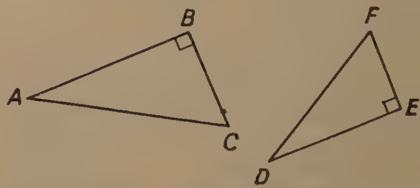
Two angles:
$\hat{B} = \hat{E}$ and $\hat{A} = \hat{D}$



Two ratios:
$\dfrac{AB}{AC} = \dfrac{DE}{DF}$ and $\dfrac{BA}{BC} = \dfrac{ED}{EF}$



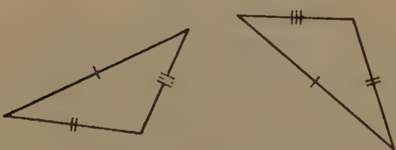
The angle and ratio at one vertex:
$\hat{B} = \hat{E}$ and $\dfrac{BA}{BC} = \dfrac{ED}{EF}$



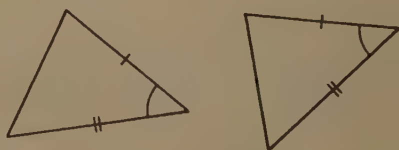
If the angle is a right-angle the ratio may be the hypotenuse and another side.
$\dfrac{BC}{BA} = \dfrac{EF}{ED}$ or $\dfrac{CB}{CA} = \dfrac{FE}{FD}$

CONGRUENT TRIANGLES

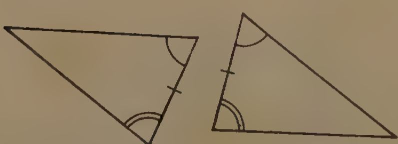
Two triangles are congruent (i.e. agree about all sides and angles) if they agree about:



(a) Three sides (*S.S.S.*)



(b) Two sides and the included angle (*S.A.S.*)



(c) Two angles and a corresponding side (*A.A.S.*)

(d) In a right-angled triangle the hypotenuse and one other side (*R.H.S.*)

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